Principle of Least action

Action of a dynamical system over an interval $t_1 < t < t_2$ is

$$A = \int_{t_1}^{t_2} 2T \, dt$$

where T = K.E.

This principle states that the variation of action along the actual path between given time interval is least, i.e.,

$$\delta \int_{t_1}^{t_2} 2T \, dt = 0 \qquad \ldots(1)$$

Now we know that T + V = E (constant)

V = P.E. and L = T−V

Scanned with CamScanner

By Hamilton's principle, we have

$$\int_{t_1}^{t_2} \delta L \; dt = 0 \;\; \text{or} \;\; \int_{t_1}^{t_2} \delta (T - V) dt = 0$$

$$\Rightarrow \quad \int_{t_1}^{t_2} \delta (T - E + T) \; dt = 0$$

$$\Rightarrow \quad \int_{t_1}^{t_2} [\delta(2T) - \delta E] \; dt = 0$$

$$\Rightarrow \quad \int_{t_1}^{t_2} \delta(2T) \, dt = 0 \qquad [\text{using } E = \text{constant} \quad \therefore \; \delta E = 0]$$

$$\Rightarrow \quad \delta \int_{t_1}^{t_2} 2T \; dt = 0$$

Distinction between Hamilton's Principle and Principle of least action:-

Hamilton's principle $\delta S = 0$ is applicable when the time interval $(t_2 - t_1)$ in passing from one configuration to the other is prescribed whereas the principle of least action i.e. $\delta A = 0$ is applicable when the total energy of system in passing from one configuration to other is prescribed and the time interval is in no way restricted. This is the essential distinction between two principles.

Scanned with CamScanner

Whittaker's Equations :- We consider a generalised conservative system i.e. an arbitrary system for which the function H is not explicitly dependent on time. For it, we have

$$H(q_j, p_j) = E_0 \text{ (constant)} \qquad ...(1)$$

where $j = 1, 2, ...n$

(2n – dimensional phase space in which q_j, p_j are coordinates)

Then basic integral invariant I will becomes

$$I = \int \left(\sum p_j \, \delta q_j - H \delta t \right)$$

$$\Rightarrow \qquad I = \int \sum_{j=1}^{n} p_j \, \delta q_j \qquad [\Theta \text{ for a conservative system, } \delta t = 0] \qquad ...(2)$$

solving (1) for one of the momenta, for example p_1.

$$p_1 = -k_1(q_1, q_2, ..., q_n, p_2 ... p_n, E_0) \qquad ...(3)$$

Scanned with CamScanner

Put the expression obtained in (2) in place of p_1, we get

$$I = \int \left[\sum_{j=2}^{n} p_j \ \delta q_j + p_1 \ \delta q_1 \right]$$

$$= \int \left[\sum_{j=2}^{n} p_j \ \delta q_j - k_1 \ \delta q_1 \right] \qquad \ldots(4)$$

But this integral invariant (4) again has the form of Poincare – Cartan integral if it is assumed that the basic co-ordinate and momenta are quantities q_j & p_j ($j = 2, 3 \ldots n$) & variable q_1 plays the role of time variable (and in place of H, we have function k_1). Therefore the motion of a generalised conservative system should satisfy the following Hamilton's system of differential equations ($2n - 2$).

$$\dot{q}_j = \frac{dq_j}{dt} = \frac{\partial k_1}{\partial p_j}; \quad \frac{-\partial k_1}{\partial q_j} = \frac{dp_j}{dq_1} \qquad j = 2, 3 \ldots. \qquad \ldots(5)$$

The equations (5) were obtaioned by Whittaker and are known as Whittaker's equations.

Scanned with CamScanner

Theorem of Lee-Hwa-Chung

If $$I' = \int \sum_{i=1}^{n} [A_i (q_k, p_k, t)\, \delta q_i + B_i(t, q_k, p_k) \delta p_i]$$

is a universal relative integral invariant, then **I′ = c I**, where c is a constant and I_1 is Poincare integral.

For $n = 1,\quad I' = \int (A\delta q + B\delta p)$

$$\Rightarrow \quad I' = c\int p\delta q = cI_1$$

$$\left[I_1 = \int \sum_{i=1}^{n} [p_i\, \delta q_i] \right]$$

and $I_1 = \int p\delta q - H\delta t$ from Poincare Cartan integral.

Scanned with CamScanner