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Notes on Paper – I, Functional Analysis
Mathematics, IV Semester (Year: 2019-2020)

Reference Book: Introductory Functional Analysis with Applications by E. Kreyszig

Unit III Content Chapter 3
Inner product spaces (Hilbert spaces)
Sections 3.1, 3.2, 3.3

All symbols have their usual meanings. We use symbols X, Y, Z for vecter spaces over a field. Usually the underlying field is either R, the field of real numbers or C, the field of complex numbers. We use symbol H for a Hilbert space; the symbol $\langle x, y\rangle$ for inner product of elements $x, y \in X$.

3.1.1 Definition An inner product on X is a mapping of $X \times X$ into the scalar field K satisfying the following properties:

1. $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle$
2. $\langle \alpha x, y\rangle = \alpha\langle x,y\rangle$
3. $\langle x,y\rangle = \overline{\langle y,x\rangle}$
4. $\langle x,x\rangle \geq 0$
5. $\langle x,x\rangle = 0 \Leftrightarrow x = 0$ (the zero vecter)

If K is the real field, then ③ reduces to $\langle x,y\rangle = \langle y,x\rangle$ called the symmetry. Otherwise an inner product is conjugate symmetric ($K = C$)

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Let $(X, < >)$ be an inner product space. ~~Define~~ For $x \in X$, define

$$\|x\| = \langle x, x \rangle$$

(This is well defined since $\langle x,x \rangle \geq 0$ for all $x \in X$)

Note that it satisfies the ~~~~ conditions of a norm on X. Hence an inner product gives rise to a norm on X. (Verify the conditions ~~Further~~ of a norm)

Further define $d : X \times X \rightarrow R$ by

$$d(x,y) = \|x-y\|.$$

Then it defines a metric on X.

To conclude, an inner product space X is a ~~~~ real or complex linear space with an inner product defined on it.

A Hilbert space is a complete inner product space.

Note that all Inner product spaces become normed linear spaces with the norm defined ~~~~ above.

All Hilbert spaces are Banach spaces.

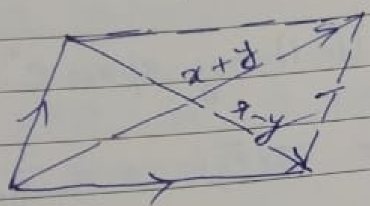
The properties of the inner product ~~~~ can be combined as

$\langle x, \alpha y + \beta z \rangle = \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle$

For an ~~[illegible]~~ inner product $(X, \langle\ \rangle)$ and $x, y \in X$,

$$\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y \rangle + \langle x-y, x-y \rangle + \langle y, x \rangle + \langle y, y \rangle$$
$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle + \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, -y \rangle$$
$$= 2\langle x, x \rangle + 2\langle y, y \rangle$$
$$= 2\|x\|^2 + 2\|y\|^2$$

This is called the parallelogram equality



Note the norm introduced with the help of an inner product satisfies the parallelogram law. Conclude that all normed linear spaces are not inner product spaces.

Examples.

1. Consider the Euclidean space R^n. For $x = (\xi_1, \xi_2, \dots \xi_n)$; $y = (\eta_1, \eta_2 \dots \eta_n)$ define $\langle x, y \rangle = \xi_1\eta_1 + \xi_2\eta_2 + \dots + \xi_n\eta_n$

$$= \sum_{i=1}^{n} \xi_i \eta_i$$

(Verify the inner product conditions)

For $x \in R^n$, $\|x\| = \langle x, x \rangle^{1/2}$
$= (\xi_1^2 + \xi_2^2 + \cdots \xi_n^2)^{1/2}$

The metric is $\|x - y\| = \langle x-y, x-y \rangle^{\frac{1}{2}}$
$= [(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + \cdots + (\xi_n - \eta_n)^2]^{1/2}$

the usual Euclidean metric on R^n.

Let $(X, \langle \, \rangle)$ be an inner product space
An element $x \in X$ is said to be orthogonal to $y \in X$
if their inner product is zero i.e. $\langle x, y \rangle = 0$.

Ex. 2. Consider the Unitary space $C^n = \underbrace{C \times C \times \cdots C}_{n \text{ times}}$

Define $\langle x, y \rangle = \xi_1 \bar{\eta}_1 + \cdots + \xi_n \bar{\eta}_n$ $x = (\xi_i)$, $y = ($

Verify that it defines an inner product on C^n.

3. Space $L^2[a, b]$: for $x, y \in L^2[a, b]$.

define $\langle x, y \rangle = \int_a^b x(t) y(t) \, dt$ for real

$\langle x, y \rangle = \int_a^b x(t) \overline{y(t)} \, dt$ for comp

Verify the conditions of an inner product; se

Ex. 4. The sequence space l^2 is a Hilbert space with inner product defined by

$$\langle x, y \rangle = \sum_{i=1}^{\infty} \xi_i \overline{\eta_i}$$

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}} = \left(\sum_{i=1}^{\infty} |\xi_i|^2 \right)^{1/2}$$

For $p \neq 2$, l^p, the sequences are not inner product spaces.

l^2 is the only inner product spaces, it was mainly studied by J. von Neumann (1927) and it is hence of importance.

Ex. 5. For $p \neq 2$, l^p is not an inner product space and hence cannot be a Hilbert space.

Let $x = (1, 1, 0, \dots) \in l^p$

$y = (1, -1, 0, \dots) \in l^p$.

Note that $\|x\| = 2^{1/p}$, $\|y\| = 2^{1/p}$.

$\|x+y\| = 2$, $\|x-y\| = 2$.

Hence the parallelogram law is not satisfied by this norm. Hence it is not introduced with the help of an ~~in~~ inner product.

Further note that l^p is complete. Hence it is a Banach space.

Conclude that not every Banach space is a Hilbert space.

Ex 6. Space $C[a,b]$

Note that $C[a,b] = \{ f : [a,b] \to \mathbb{R} \text{ (or } \mathbb{C}) \mid f \text{ is continuous} \}$

With the usual addition and scalar multiplication of functions, $C[a,b]$ is a vector space.

The norm of a function $f \in C[a,b]$ is defined as
$$\|f\| = \max_{t \in [a,b]} |f(t)|$$

We observe that this norm is not induced with the help of an inner product. To prove this assertion we just see that the law of parallelogram is not satisfied by this norm:

Take $x(t) = 1$ for all $t \in [a,b]$ and
$$y(t) = \frac{(t-a)}{b-a} \quad \text{for } t \in [a,b].$$

Then both $x(t), y(t) \in C[a,b]$.

Also
$$x(t) + y(t) = 1 + \frac{t-a}{b-a}$$
$$x(t) - y(t) = 1 - \frac{t-a}{b-a}$$

$\|x+y\| = 2$, $\|x-y\| = 1$ and

$\|x+y\|^2 + \|x-y\|^2 = 5$ while $2(\|x\|^2 + \|y\|^2) =$

3.2 Further Properties of Inner Product Spaces

3.2.1 Schwarz inequality

For an innerproduct space $(X, < >)$ we have

$$|<x, y>| \leq \|x\| \|y\|$$

where the equality holds if and only if $\{x, y\}$ is a linearly dependent set.

Proof. If $y=0$, then the inequality holds since $<x,0>=0$

Take $y \neq 0$. For any scalar α we have

$$0 \leq \|x-\alpha y\|^2 = <x-\alpha y, x-\alpha y>$$

$$= <x,x> - \bar{\alpha} <x,y> - \alpha [<y,x> - \bar{\alpha} <y,y>]$$

If we choose $\bar{\alpha} = \dfrac{<y,x>}{<y,y>}$, then the expression in the bracket is zero. The remaining relation is

$$0 \leq <x,x> - \frac{<y,x>}{<y,y>} <x,y> = \|x\|^2 - \frac{|<x,y>|^2}{\|y\|^2}$$

Here: $<y,x> = \overline{<x,y>}$. Multiplying whole thing by $\|y\|^2$ and solving we get

$$|<x,y>|^2 \leq \|x\|^2 \|y\|^2$$

or $$|<x,y>| \leq \|x\| \|y\|$$

In this relation equality holds if and only if $y=0$ or $x = \alpha y$ which shows linear dependence.

6. Triangle inequality

The norm $||x|| = \sqrt{\langle x, x\rangle}$ in an inner product space X satisfies the triangle inequality i.e.

$$||x+y|| \leq ||x|| + ||y|| \quad \text{for } x, y \in X.$$

the equality sign holds iff $y = 0$ or $x = cy$.

We have

$$||x+y|| = \langle x+y, x+y\rangle$$
$$= ||x||^2 + \langle x, y\rangle + \langle y, x\rangle + ||y||^2$$

By using the Schwarz inequality,

$$|\langle x, y\rangle| = |\langle y, x\rangle| \leq ||x||\,||y||$$

By triangle inequality for numbers, we obtain

$$||x+y||^2 \leq ||x||^2 + 2|\langle x, y\rangle| + ||y||^2$$
$$\leq ||x||^2 + 2||x||\,||y|| + ||y||^2$$
$$= (||x|| + ||y||)^2$$

or $$||x+y|| \leq ||x|| + ||y||.$$

3.2.2 Lemma Inner product is continuous i.e. if $x_n \longrightarrow x$, $y_n \longrightarrow y$, then

$$\langle x_n, y_n\rangle \longrightarrow \langle x, y\rangle$$

Proof. Note that

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| = |\langle x_n, x\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle|$$

$$\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle|$$

$$\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\| \to 0$$

since $y_n \to y$ and $x_n \to x$ as $n \to \infty$.

Hence inner product is continuous.

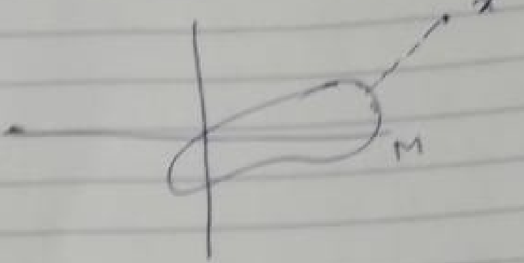
3.2.3 Theorem. For any inner product space X there exists a Hilbert space H and an isomorphism A from X onto a dense subspace $W \subset H$. The space H is unique upto isomorphism.

Proof. See proof in Kreyszig's book. Ask problems if you have difficulties.

3.3 Orthogonal Complements and Direct Sums

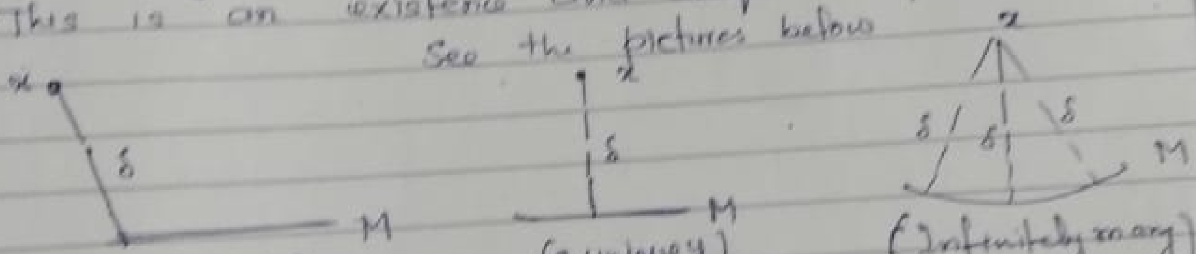
For a metric space (X,d) and a subset $M \subset X$, the distance of a point $x \in X$ and the set M is defined to be

$$\delta = \inf_{y \in M} d(x,y)$$



Ex. in $\mathbb{R}^2$

A fundamental question is ? there is a point $y \in M$ and moreover this y is unique satisfying $\delta = d(x,y)$ and moreover this y is unique. This is an existence and uniqueness problem. See the pictures below



(No y) (a unique y) (Infinitely many y's)

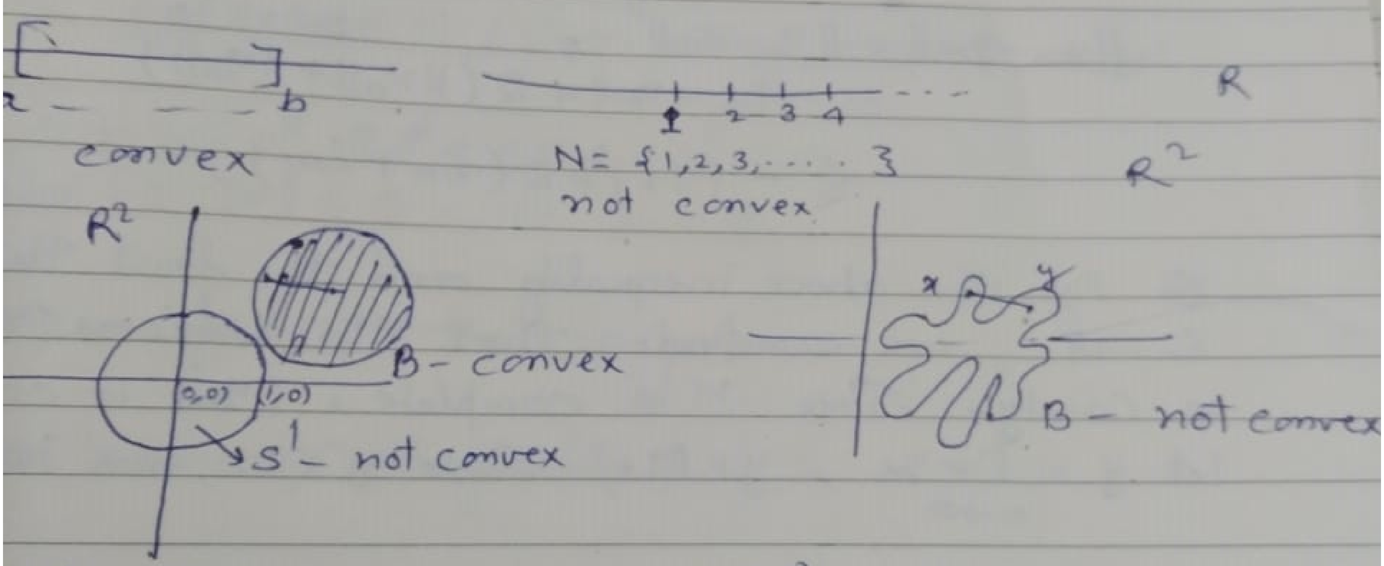
In the first result of this section the existence and uniqueness problem is discussed in an inner product space.

First we need the following:
For a vector space X and $x, y \in X$, the line

segment joining x and y is defined to be the set of all $z \in X$ of the form

$$z = \alpha x + (1-\alpha) y; \ \alpha \in R, \ 0 \le \alpha \le 1$$

A subset $M \subset X$ is called a convex set if for each pair of points $x, y \in M$ the segment joining x and y lies entirely in M, see figures given below



(Make more example)

3.1 Theorem. Let X be an inner product space and $\ne \phi$ be a convex set which is complete. Then for any given $x \in X$ there exists a unique $y \in M$ such that

$$\delta = \inf_{\tilde{y} \in M} \| x - \tilde{y} \| = \| x - y \|$$

Pf. By using the definition of infimum we get a sequence $\delta_n \to \delta$, where $\delta_n = \| x - y_n \|$, $y_n \in M$

We not that the sequence (y_n) is Cauchy in M.
Take $y_n - x = v_n$. Then $\|v_n\| = \delta_n$ and

$$\|v_n + v_m\| = \|y_n + y_m - 2x\| = 2\|\tfrac{1}{2}(y_n + y_m - x)\| \geq 2\delta.$$

Since M is given to be convex and y_n, y_m are elements of M, $\frac{1}{2}(y_n + y_m) \in M$. Moreover $y_n - y_m = v_n - v_m$

By using the parallelogram law for inner product spaces we have

$$\|y_n - y_m\|^2 = \|v_n - v_m\|^2$$
$$= -\|v_n + v_m\|^2 + 2(\|v_n\|^2 + \|v_m\|^2)$$
$$\leq -(2\delta)^2 + 2(\delta_n^2 + \delta_m^2),$$

By the above inequality and the fact that $\delta_n \to \delta$, we conclude that the sequence (y_n) is Cauchy. Since M is complete, (y_n) is convergent. Let $y = \lim_{n \to \infty} y_n$, $y \in M$. Since $y \in M$, we have

$\|x - y\| \geq \delta$. Moreover,

$$\|x - y\| \leq \|x - y_n\| + \|y_n - y\|$$
$$= \delta_n + \|y_n - y\| \longrightarrow \delta.$$

This shows that $\|x - y\| = \delta$

In the next step we see that the point y is unique. If possible assume that $\exists$ another point

$y_0 \in M$ and it satisfies

$$\|x - y_0\| = \delta \quad \text{alongwith } \|x-y\| = \delta.$$

By the parallelogram law

$$\|y-y_0\|^2 = \|(y-x)-(y_0-x)\|^2$$
$$= 2\|y-x\|^2 + 2\|y_0-x\|^2 - \|(y-x)+(y_0-x)\|^2$$
$$= 2\delta^2 + 2\delta^2 - 2\left\|\tfrac{1}{2}(y+y_0) - x\right\|^2$$

Since $\frac{1}{2}(y+y_0) \in M$ (M being convex) , we get

$$\|y-y_0\|^2 = 2\delta^2 + 2\delta^2 - 4\delta^2 = 0$$

Hence $\|y - y_0\| \leq 0$. or $y_0 = y$.

This proves the uniqueness.

3.3. 2 Lemma (Orthogonality).

In the previous theorem, let M be a complete subspace Y and $x \in X$ be fixed. Then $z = x - y$ is orthogonal to Y.

Proof. To prove the result we use the method of contradiction. If possible, assume that z is not orthogonal to Y. Then there is a point (say) $y_1 \in Y$ satisfying $\langle z, y_1 \rangle = \beta \neq 0$

(14)

Note that $y_1 \neq 0$ since otherwise $\langle z, y_1 \rangle = 0$.
For any scalar α,

$$\|z-\alpha y_1\|^2 = \langle z-\alpha y_1, z-\alpha y_1 \rangle$$
$$= \langle z,z\rangle - \bar{\alpha}\langle z,y_1\rangle - \alpha\left[\langle y_1,z\rangle - \bar{\alpha}\langle y_1,y_1\rangle\right]$$
$$= \langle z,z\rangle - \bar{\alpha}\beta - \alpha\left[\bar{\beta} - \bar{\alpha}\langle y_1,y_1\rangle\right]$$

If we take $\bar{\alpha} = \dfrac{\bar{\beta}}{\langle y_1, y_1\rangle}$, the term in the bracket becomes zero.

From theorem 3.3.1 we have

$\|z\| = \|x-y\| = \delta$, so that our equation becomes

$$\|z-\alpha y_1\|^2 = \|z\|^2 - \frac{|\beta|^2}{\langle y_1, y_1\rangle} < \delta^2$$

which is not possible, since

$z-\alpha y_1 = x - y_1 - \alpha y_1$ with $y-\alpha y_1 \in Y$

so that $\|z-\alpha y_1\| \geq \delta$ by definition of δ.

This completes the proof.

3.3 Definition. A vector space X is said to be the direct sum of two subspaces Y and Z of X, written

$$X = Y \oplus Z$$

if each $x \in X$ has a unique representation
$$x = y + z \quad , \quad y \in Y, z \in Z.$$

In this case the subspace Z is called an algebraic complement of Y in X and vice versa.

The subspaces Y and Z are called complementary pair of subspaces in X.

Example. $Y = R$ is a subspace of the Euclidean plane R^2. Then Y has infinitely many algebraic complements. The most convenient is a complement that is perpendicular.

Definition. Let H be a Hilbert space and Y be a closed subspace of X. Then the orthogonal complement of Y is the following

$$Y^{\perp} = \{z \in H \mid z \perp y\}$$
$$= \{z \in H \mid \langle y, z\rangle = 0\}$$

ie. $Y^{\perp}$ is the set of all vectors which are orthogonal to Y. Note that for a Hilbert space H,
$$\{0\}^{\perp} = H \quad \text{and} \quad H^{\perp} = \{0\}.$$

3.3.4 Theorem. Let Y be any closed subspace of a Hilbert space H. Then
$$H = Y \oplus Z \qquad Z = Y^{\perp}$$

Proof. Note that being a Hilbert space, H is complete. Since Y is a closed subspace of H, it is also complete. Being a subspace, Y is a convex set of H. Combining the results of Theorem 3.3.1 (give statements) and Lemma 3.3.2 (give statements) we obtain that for every point $x \in H$ there is a $y \in Y$ such that

$$x = y + z \quad , z \in Z = Y^{\perp}.$$

Next we show that the above representation is unique. If possible, assume that

$$x = y + z = y_1 + z_1$$

where $y, y_1 \in Y$ and $z, z_1 \in Z$. Then

$$y - y_1 = z_1 - z$$

Since $y - y_1 \in Y$ and $z_1 - z \in Z = Y^{\perp}$, we see that

$$y - y_1 \in Y \cap Y^{\perp} = \{0\}$$

It follows that $y = y_1$. Hence $z_1 = z$

ie the representation is unique.

The above discussion motivates to the concept of projection mapping onto a subspace of a Hilbert space This leads to the following mapping

$$P : H \longrightarrow Y \quad \text{taking } x \in H \text{ to } y = Px.$$

since for $x \in H$, y is unique, the mapping P is well defined.

P is called the projection of H onto Y.

P has the following properties.

a. P maps H onto Y

b. Y onto itself

c. $Z = Y^{\perp}$ onto $\{0\}$.

Also $P^2 = P \circ P = P$ i.e. P is idempotent.

P/Y is the identity.

For the subspace Z we have the following lemma

3.3.5 Lemma (Null space). The orthogonal complem Y$^{\perp}$ of a closed subspace Y of a Hilbert space is the null space $\mathcal{N}(P)$ of the orthogonal projec P of H onto Y.

Note that $\mathcal{N}(P) = \{x \in H \mid P(x) = 0\}$

$= \cancel{\{x \in H \mid P(y+z)\}}$

$= Y^{\perp}$ (easy to see)

Definition For an inner product space X and a subset M of X, the set

$$M^{\perp} = \{x \in X \mid x \perp M\}$$

$$= \{x \in X \mid \langle x, v \rangle = 0 \ \forall\, v \in$$

is called the annihilator $M^{\perp}$ of the set M.

The annihilator has the following properties:

$M^{\perp}$ is a subspace of X:

Let $x, y \in M^{\perp}$ and α, β be scalars. Then for a point $v \in M$,

$$\langle \alpha x + \beta y, v \rangle = \alpha \langle x, v \rangle + \beta \langle y, v \rangle = 0$$

Hence $\alpha x + \beta y \in M^{\perp}$ $\because$ v is arbitray.

Also $M^{\perp}$ is closed (Prove)

Moreover $M \subset (M^{\perp})^{\perp}$ ie. $M \subset M^{\perp\perp}$

For $x \in M$, $x \perp M^{\perp}$ ie. $x \in (M^{\perp})^{\perp}$

For closed subspaces we have the following

3.3.6 Lemma (Closed Subspaces). If Y is a closed subspace of a Hilbert space H, then

$$Y = Y^{\perp\perp}$$

Proof. For the above discussion, we already have

$Y \subset Y^{\perp\perp}$. To prove the reverse ~~inequa~~ containment, ie to show that $Y^{\perp\perp} \subset Y$, take a point $x \in Y^{\perp\perp}$. By using the result 3.3.4, we have a unique representation $x = y + z$, where $y \in Y \subset Y^{\perp\perp}$.

Since $x \in Y^{\perp\perp}$, $y \in Y^{\perp\perp}$, we have

$$x - y = z \text{ (say) } \in Y^{\perp\perp} \quad (\because Y^{\perp\perp} \text{ is a subspace})$$

It means that $z \perp Y^{\perp}$. Also $z \in Y^{\perp}$

Combining we get $\langle z, z\rangle = 0$ or $z = 0$.

Meaning thereby $x = y$ or $x \in Y$.

Since $x \in Y^{\perp\perp}$ is ~~arbitrg~~ arbitrary, $Y^{\perp\perp} \subset Y$

3.3.7. Lemma For a subset $M \neq \phi$ of a Hilbert H; the span of M is dense in H if and only if $M^{\perp} = \{0\}$.

Proof Left for students.