UNIT - V (Continue)...

Paper - Lebegue Measure & Integration

Book - G.D. Barra,
Lebesgue Measure & integration

Facutly - Dr. Pradeep Porwal

Almost Uniform Convergence):-

A Sequence $\{f_n\}$ of measurable func^n is called converges almost Uniformly to a measurable func^n f if for each $\epsilon > 0$ $\exists$ a measurable set $A \subset E$ with

$m(A) < \epsilon$ Such that

$\{f_n\}$ converges to f uniformly on $(E-A)$

It is denoted by $f_n \to f$ a.u.

If $f_n \to f$ a.u. then $f_n \to f$ a.e.

Suppose that $f_n \to f$ a.u.

then by def^n

for each positive integer $n \in N$ $\exists$ a measurable set such that

$$m(E_n) < \frac{1}{n} \quad \&$$

$$f_n \to f \text{ uniformly on } (E - E_n)$$

Set $A = \bigcup_{n=1}^{\infty} (E - E_n)$

Then $m(E-A) = m\left[E - \bigcup_{n=1}^{\infty}(E-E_n)\right]$

$$= m\left(E - E \bigcap_{n=1}^{\infty} E_n\right)$$

$$= m\left(\bigcap_{n=1}^{\infty} E_n\right)$$

$\leq m(E_n)$

$m(E-A) \leq \frac{1}{n}$

Taking limit $n \to \infty$

$$\lim_{n\to\infty} m(E-A) \leq \lim_{n\to\infty} \frac{1}{n}$$

$$m(E-A) \leq 0$$

But

$$m(E-A) \geq 0$$

Hence

$m(E-A) = 0$ for each $x \in A$

$\Rightarrow f_n \to f$ a.e. {by defn}

If $f_n \to f$ a.u. then $f_n \to f$ in measure.

Suppose on the contrary that

$f_n \not\to f$ in measure then by defn for every $\epsilon > 0$ and $\delta > 0$ such that

$$\lim \mu[x : |f_n(x) - f(x)| \geq \epsilon] > \delta \quad \text{--- (1)}$$

$f_n \to f$ a.u. then by defn $\exists$ a set E with $m(E) < \delta$ such that

$f_n \to f$ uniformly on E^c

This is contradiction of eq^n (1)

Hence $f_n \to f$ in measure.

If $f_n \to f$ in measure then sequence $\{f_n\}$ is a Cauchy sequence in measure.

Assume that $f_n \to f$ in measure then by def^n for every $\epsilon > 0$:

Such that

$$\lim \mu[x : |f_n(x) - f(x)| > \epsilon] = 0 \quad -(1)$$

Consider

$$|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)|$$

$$\leq |f_n(x) - f(x)| + |f(x) - f_m(x)|$$

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)|$$

$$[x : |f_n(x) - f_m(x)| > 2\epsilon] \subseteq [x : |f_n(x) - f(x)| > \epsilon]$$

$$\cup [x : |f_m(x) - f(x)| > \epsilon]$$

Taking measure

$$\mu[x : |f_n(x) - f_m(x)| > 2\epsilon] \leq \mu[x : |f_n(x) - f(x)| > \epsilon] + \mu[x : |f_m(x) - f(x)| > \epsilon]$$

Taking lim $n \to \infty$

$$\lim \mu[x : |f_n(x) - f_m(x)| \geq 2\varepsilon] \leq \lim \mu[x : |f_n(x) - f(x)| \geq \varepsilon] + \lim \mu[x : |f_m(x) - f(x)| \geq \varepsilon]$$

$$\lim \mu[x : |f_n(x) - f_m(x)| \geq 2\varepsilon] \leq 0 + 0 \quad \} \text{ by } 1$$

$\because$ But $\lim \mu[x : |f_n(x) - f_m(x)| \geq 2\varepsilon] \geq 0$

Hence

$$\lim \mu[x : |f_n(x) - f_m(x)| \geq 2\varepsilon] = 0$$

$\Rightarrow \{f_n\}$ is a Cauchy sequence in measure {by def^n}

1. Let d is metric where $d : L^p(\mu) \times L^p(\mu) \to R$ is defined by

$$d(f, g) = \|f g\|_p = \left(\int_X |f - g|^p \, d\mu\right)^{1/p} \quad \forall f, g \in L^p(\mu)$$

Show that (X, d) is metric space.

Given that

$$d(f, g) = \|f g\|_p = \left(\int_X |f - g|^p \, d\mu\right)^{1/p} \quad \text{--- (1)}$$

$$\forall f, g \in L^p(\mu)$$

(i). By eqⁿ (1) non negativity

$$d(f,g) = \|f-g\|_p = \left(\int_X |f-g|^p \, d\mu\right)^{1/p} \geq 0$$

$\{\because |x| \geq 0\}$

$d(f,g) \geq 0$

$\therefore$ (i) hold's.

(ii) Let $d(f,g) = 0 \Leftrightarrow \left(\int_X |f-g|^p \, d\mu\right)^{1/p}$

$\Leftrightarrow 0$

$\Leftrightarrow |f-g|^p = 0$

$\Leftrightarrow |f-g| = 0$

$f - g = 0$

ie $d(f,g) = 0 \Leftrightarrow f = g$

$\therefore$ (ii) hold.

(iii). by (1) $d(f,g) = \|f-g\|_p = \left(\int_X |f-g|^p \, d\mu\right)^{1/p}$

$$d(f,g) = \left(\int_X |g-f|^p \, d\mu\right)^{1/p}$$

$= \|g-f\|_p$ {by (1)

$d(f,g) = d(g,f)$ {by (1)

(iii) holds.

(iv). Consider $d(f,g) = \|f-g\|_p$

$= \|f-h+h-g\|_p$

$\leq \|f-h\|_p + \|h-g\|_p$

{by minkowski inequality

$d(f,g) \leq d(f,h) + d(h,g)$ {by (1)

(iv). holds.

Hence (X,d) is a metric space {by defn

Fundamental in measure :-

A Sequence of fun^n is called fundamental with respect to a particular kind of convergence if it forms a Cauchy Sequence

ie A Sequence $\{f_n\}$ is fundamental in measure

if for every $\varepsilon > 0$ we have

$$\lim_{m,n \to \infty} \mu[x : |f_n(x) - f_m(x)| \geq \varepsilon] = 0$$

If $\{f_n\}$ is a sequence of measurable fun^n which is fundamental in measure then $\exists$ a measurable fun^n "f" such that

$f_n \to f$ in measure.

$\because \{f_n\}$ is fundamental in measure then $\therefore$ by def^n for every possitive integer 'K' such that

$$\lim_{n,p \to \infty} \mu\left[x : |f_n(x) - f_p(x)| > \frac{1}{2^K}\right] < \frac{1}{2^K} \quad - (1)$$

Assume that $n_{K+1} > n_K$ for each "K"

This gives a sequence $\{f_{n_K}\}$ is an infinite subsequence of $\{f_n\}$

Let $E_K = \left\{x : |f_{n_K}(x) - f_{n_{K+1}}(x)| > \frac{1}{2^K}\right\} \quad - (2)$

Then for each $x \notin \bigcup_{i=K}^{\infty} E_i$; we have

$$|f_{n_i}(x) - f_{n_j}(x)| \leq \sum_{p=i}^{\infty} |f_{n_p}(x) - f_{n_{p+1}}(x)| < \frac{1}{2^{i-1}}$$

(3)

$\{j > i > p\}$

Hence the Sequence $\{f_{n_K}\}$ is a uniformly Cauchy Sequence in $\left(\bigcup_{i=K}^{\infty} E_i\right)^c$

Let $f(x) = \lim_{K\to\infty} f_{n_K}(x)$ —— (4)

Consider

$$|f_n(x) - f(x)| = |f_n(x) - f_{n_K}(x) + f_{n_K}(x) - f(x)|$$

$$|f_n(x) - f(x)| \leq |f_n(x) - f_{n_K}(x)| + |f_{n_K}(x) - f(x)|$$

$$\left[x : |f_n(x) - f(x)| \geq \varepsilon\right] \subseteq \left[x : |f_n(x) - f_{n_K}(x)| \geq \frac{\varepsilon}{2}\right] \cup \left[x : |f_{n_K}(x) - f(x)| \geq \frac{\varepsilon}{2}\right]$$

Then

$$\mu\left[x : |f_n(x) - f(x)| \geq \varepsilon\right] \leq \mu\left[x : |f_n(x) - f_{n_K}(x)| \geq \frac{\varepsilon}{2}\right] + \mu\left[x : |f_{n_K}(x) - f(x)| \geq \frac{\varepsilon}{2}\right]$$ —— (5).

$\{\because A \subseteq B \Rightarrow \mu(A) \leq \mu(B)\}$

taking limit $n\to\infty$ the R.H.S. of eq^n (5) goes to 0 by (1) & (4).

ie $\lim_{n\to\infty} \mu\left[x : |f_n(x) - f(x)| \geq \varepsilon\right] \leq 0$

But

$$\lim_{n\to\infty} \mu[x : |f_n(x) - f(x)| \geq \epsilon] \geq 0$$

So

$$\lim_{n\to\infty} \mu[x : |f_n(x) - f(x)| \geq \epsilon] = 0$$

$\Rightarrow f_n \to f$ is measure {by def^n

Que: If $f_n \to f$ in measure and $g_n \to g$ in measure then $\alpha f_n + \beta g_n \to \alpha f + \beta g$ in measure

proof Given that
$f_n \to f$ in measure & $g_n \to g$ in measure
then by def^n for every positive ϵ we have.

$$\lim_{n\to\infty} \mu[x : |f_n(x) - f(x)| \geq \epsilon] = 0 \quad - (1)$$

& $$\lim_{n\to\infty} \mu[x : |g_n(x) - g(x)| \geq \epsilon] = 0 \quad - (2)$$

Now we will show that

$$\alpha f_n + \beta g_n \to \alpha f + \beta g$$

$\because |(\alpha f_n + \beta g_n) - (\alpha f + \beta g)| = |(\alpha f_n + \beta g_n) - \{\alpha f - \beta g\}|$

$|\alpha f_n + \beta g_n - (\alpha f + \beta g)| = |(\alpha f_n - \alpha f) + (\beta g_n - \beta g)|$

$= |\alpha(f_n - f) + \beta(g_n - g)|$

$|(\alpha f_n + \beta g_n) - (\alpha f + \beta g)| \leq |\alpha(f_n - f)| + |\beta(g_n - g)|$

Taking measure both side.

$\mu[x : |(\alpha f_n + \beta g_n) - (\alpha f + \beta g)| > 2\varepsilon] \leq \mu[x : |\alpha(f_n - f)| > \varepsilon] + \mu[x : |\beta(g_n - g)| > \varepsilon]$

Taking lim $n \to \infty$ both Side

$\lim_{n\to\infty} \mu[x : |(\alpha f_n + \beta g_n) - (\alpha f + \beta g)| > 2\varepsilon] \leq$

$\lim_{n\to\infty} \mu[x : |(f_n(x) - f)| > \frac{\varepsilon}{\alpha}] + \lim_{n\to\infty} \mu[x : |g_n(x) - g(x)| > \frac{\varepsilon}{\beta}]$

$\lim_{n\to\infty} \mu[x : |(\alpha f_n + \beta g_n) - (\alpha f + \beta g)| > 2\varepsilon] \leq$

$\lim_{n\to\infty} \mu[x : |f_n(x) - f(x)| > \varepsilon'] + \lim_{n\to\infty} \mu[x : |g_n(x) - g(x)| > \varepsilon'']$

$\leq 0 + 0 \quad \{$ by 1 & 2

$\lim_{n\to\infty} \mu[x : |(\alpha f_n + \beta g_n) - (\alpha f + \beta g)| > 2\varepsilon \to 0$

Hence

$$\lim_{n\to\infty} \left[x : |(\alpha f_n + \beta g_n) - (\alpha f + \beta g)| > 2\varepsilon \right] = 0$$

$$\alpha f_n + \beta g_n = \alpha f + \beta g \quad \text{a.e.}$$

by def^n

Hence

$(\alpha f_n + \beta g_n) \to (\alpha f + \beta g)$ in measure.