UNIT - V

Paper - Lebegue Measure & Integration

Book - G. D. Barra,
Lebesgue Measure & integration

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Convergence in Measure :-

Let $\{f_n\}$ be a sequence of measurable fun^n and f is a measurable fun^n

Then

$f_n \to f$ in measure if for every possitive we have

$$\lim_{n\to\infty} \mu[x : |f_n(x) - f(x)| > \varepsilon] = 0$$

A sequence of measurable fun^n convergess in measure then the limit fun^n is unique almost everywhere.

Let $f_n \to f$ in measure and $f_n \to g$ in measure

then by def^n "for every possitive ε" we have

$$\lim_{n\to\infty} \mu[x : |f_n(x) - f(x)| > \varepsilon] = 0 \quad \text{--- (1)}$$

$$\lim_{n\to\infty} \mu[x : |f_n(x) - g(x)| > \varepsilon] = 0 \quad \text{--- (2)}$$

Now we show that $f = g$ a.e.

$\because |f-g| = |f - f_n + f_n - g|$

$\leq |f - f_n| + |f_n - g|$

$|f-g| \leq |f_n - f| + |f_n - g|$

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then

$$[x : |f(x) - g(x)| \geq 2\epsilon] \subseteq [x : |f_n(x) - f(x)| \geq \epsilon]$$
$$\cup [x : |f_n(x) - g(x)| \geq \epsilon]$$

Taking measure on both side

$$\mu[x : |f(x) - g(x)| \geq 2\epsilon] \leq \mu[x : |f_n(x) - f(x)| \geq \epsilon]$$
$$+ \mu[x : |f_n(x) - g(x)| \geq \epsilon]$$

$$\left\{ \begin{array}{l} \because A \subseteq B \Rightarrow m^*(A) \leq m^*(B) \\ m^*(A \cup B) \leq m^*(A) + m^*(B) \end{array} \right.$$

Taking lim $n \to \infty$

$$\lim_{n\to\infty} \mu[x : |f(x) - g(x)| \geq 2\epsilon] \leq \lim_{n\to\infty} \mu[x : |f_n(x) - f(x)| \geq \epsilon] + \lim_{n\to\infty} \mu[x : |f_n(x) - g(x)| \geq \epsilon]$$

$$\lim_{n\to\infty} \mu[x : |f(x) - g(x)| \geq 2\epsilon] \leq 0 + 0 \quad \{\text{by } ① \,\&\, ②$$

$$\lim_{n\to\infty} \mu[x : |f(x) - g(x)| \geq 2\epsilon] \leq 0$$

$$\lim_{n\to\infty} \mu[x : |f(x) - g(x)| \geq 2\epsilon] \geq 0$$

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$$\lim_{n\to\infty} \mu\left[x : |f(x) - g(x)| > 2\epsilon\right] = 0$$

$$f = g \text{ a.e.} \qquad \{\text{by def}^n\}$$

Hence the limit fun^n is unique a.e.

Show that if $f_n \to f$ in measure and α is any scalar then $\alpha f_n \to \alpha . f$ in measure.

Let

$f_n \to f$ in measure

Then by def^n for every possitive ϵ we have

$$\lim_{n\to\infty} \mu\left[x : |f_n(x) - f(x)| > \epsilon\right] = 0 \quad -(1)$$

Now we have Show that $\alpha f_n = \alpha f$ a.e.

$$\because \; |\alpha f_n - \alpha f| = |\alpha(f_n - f)|$$

$$\left[x : |\alpha f_n(x) - \alpha f(x)| > \epsilon\right] = \left[x : \alpha |f_n(x) - f(x)| > \epsilon\right]$$

$$\left[x : |\alpha f_n(x) - \alpha f(x)| > \epsilon\right] = \left[x : |f_n(x) - f(x)| > \frac{\epsilon}{\alpha}\right]$$

Taking measure on both Side

$$\mu\left[x : |\alpha f_n(x) - \alpha f(x)| > \epsilon\right] = \mu\left[x : |f_n(x) - f(x)| > \frac{\epsilon}{\alpha}\right]$$

taking $\lim n \to \infty$

$$\lim_{n\to\infty} \mu\left[x : |\alpha f_n(x) - \alpha f(x)| > \epsilon\right] = \lim_{n\to\infty} \mu\left[x : |f_n(x) - f(x)| > \frac{\epsilon}{\alpha}\right]$$

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$$\lim_{n\to\infty} \mu\left[x : |\alpha f_n(x) - \alpha f(x)| > \varepsilon\right] = \lim_{n\to\infty} \mu\left[x : |f_n(x) - f(x)| > \varepsilon\right]$$

$$\left\{ \because \frac{\varepsilon}{\alpha} = \varepsilon' \right\}$$

$$\lim_{n\to\infty} \mu\left[x : |\alpha f_n(x) - \alpha f(x)| > \varepsilon\right] = 0$$

$$\alpha f_n = \alpha f \quad a.e.$$

$$\{\text{by def}^n\}$$

Show that if $f_n \to f$ in measure and $g_n \to g$ in measure that then

$$f_n + g_n \to f + g$$

Given let $f_n \to f$ and $g_n \to g$ in measure

Then by def^n for every positive ε we have

$$\lim_{n\to\infty} \mu\left[x : |f_n(x) - g(x)| > \varepsilon\right] = 0 \quad \text{--- (1)}$$

$$\lim_{n\to\infty} \mu\left[x : |g_n(x) - g(x)| > \varepsilon\right] = 0 \quad \text{--- (2)}$$

Now we will show that

$$f_n + g_n = f + g \quad a.e.$$

$$\because |(f_n + g_n) - (f + g)| = |(f_n + g_n) + (-f - g)|$$

$$|(f_n + g_n) - (f + g)| = |f_n - f| + |g_n - g|$$

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hen

$[x : |(f_n+g_n) - (f+g)| > 2\varepsilon] \subseteq [x : |f_n(x) - f(x)|$

$> \varepsilon] \cup [x : |g_n(x) - g(x)| > \varepsilon]$

Taking measure on both side

$\lim_{n\to\infty} \mu[x : |(f_n+g_n) - (f+g)| > 2\varepsilon] \leq \lim_{n\to\infty}$

$\mu[x : |f_n(x) - f(x)| > \varepsilon] + \lim_{n\to\infty} \mu[x : |g_n(x) - g(x)| > \varepsilon]$

$\leq 0 + 0$ {by (1) & (2)}

ie.

$\lim_{n\to\infty} \mu[x : |(f_n+g_n) - (f+g)| > 2\varepsilon] \leq 0$

ut. $\lim_{n\to\infty} \mu[x : |(f_n+g_n) - (f+g)| > 2\varepsilon] \geq 0$

ence $\lim_{n\to\infty} \mu[x : |(f_n+g_n) - (f+g)| > 2\varepsilon] = 0$

$(f_n+g_n) = (f+g)$ {by def^n of a.e.}

Hence. (f_n+g_n) converges to $f+g$ in measure

Convergence in $L^p(\mu)$:-

A Sequence of a fun^n $f(n) \to f$ in the mean of order $p>0$ if

$\lim \|f_n - f\|_p = 0$

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$\|f_n - f\|_p < \varepsilon$. of $p = 1$ then

f_n is said to be converges to f in the mean.

If $f_n \to f$ in the mean of order p then $f_n \to f$ in measure.

Suppose on the contrary that

$f_n \not\to f$ ie f_n does not converges to f in measure

Then by def^n

for every ε

$$\lim \mu [x : |f_n(x) - f(x)| > \varepsilon] > \delta \quad - (1)$$

$$\|f_n - f\|_p = \left(\int |f_n - f|^p \, d\mu\right)^{1/p}$$

$$= \left(|f_n - f|^p\right)^{1/p} \left(\int d\mu\right)^{1/p}$$

$$= |f_n - f| \, \mu^{1/p}$$

$\|f_n - f\|_p < \varepsilon$

This is contradiction because f_n convergence to f in the mean of order p

Hence $f_n \to f$ in measure.

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