The velocity of sound v_s is given by $(\omega_m a/2)$.
To relate the interatomic force constant α to the Young's modulus Y $(\omega = v_s q)$. Using eqn (7)

$$v_s = \sqrt{\frac{Y}{\rho}} \quad \text{or} \quad Y = \rho v_s^2 = \frac{\rho \omega_m^2 a^2}{4} \qquad \left(\omega_m = \sqrt{\frac{4\alpha}{M}}\right.$$

$$\therefore \quad Y = \frac{\rho a^2}{4}\left(\frac{4\alpha}{M}\right) = \left(\frac{\rho a^2}{M}\right)\alpha$$

$$\boxed{\rho = M/a^3}$$

$$\therefore \quad \boxed{\alpha = aY} \qquad \text{——— (8)}$$

<u>A useful relation for estimating α</u>

$[\alpha = 5 \times 10^3 \text{ dynes/cm}]$.

As q increases, the dispersion curve begins to deviate from the straight line and tends downwards. Eventually the curve saturates at $q = \pi/a$ with a maximum frequency equal to ω_m, which ~~we~~ is found to be

$$\omega_m = \left(\frac{4\alpha}{M}\right)^{1/2} \qquad \text{——— (9)}$$

The dependance of this frequency on the force constant and the atomic mass is as one would expect for a harmonic oscillator. This result is for the behaviour of the dispersion curve in the range $0 < q < \pi/a$.

Scanned with CamScanner

For small q, $\lambda \gg a$ and the atoms move essentially in phase with eachother as indicated below:-

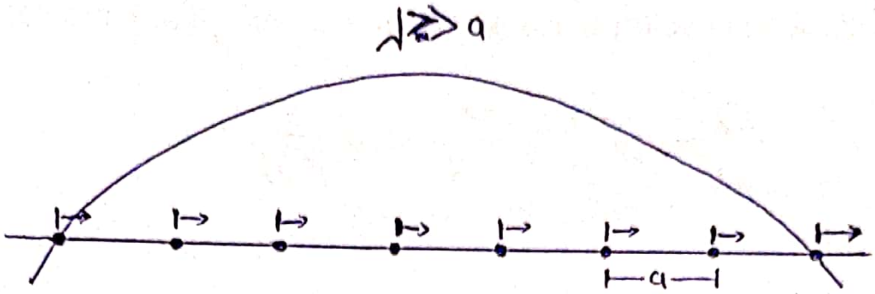

Fig 3

The restoring force on the atom due to its neighbors is therefore small, which is the reason why 'ω' is also small. In fact for $q=0$, $\lambda=\infty$ and the whole lattic moves as a rigid body, which results in the vanishing of the restoring force. This explains why $\omega=0$ at $q=0$.

The opposite limit occurs at $q=\pi/a$ where $\lambda=2a$.

Fig. 4

It is seen from the figure that the neighboring atoms are out of phase, and consequently the restoring force and the forequency are at a maximum.

By introducing a force constant which is q-dependent, we relate eq^n (2) and substitute $U_{n\pm1} = e^{\pm iqa}$, as follows from eq^n (4), eq^n (2) reduces to.

$$M\frac{d^2U_\alpha}{dt^2} = -\alpha(2U_n - U_{n+1} - U_{n-1}) \quad (2)$$

$$M\frac{d^2U_n}{dt^2} = -[4\alpha\sin^2(qa/2)]U_n \quad (10)$$

(*)

Which corresponds to a harmonic oscillator of force constant

$$\boxed{\alpha(q) = 4\alpha\sin^2(qa/2)} \quad (11)$$

This force constant depends on q or λ, because the motions of the atoms are correlated. The frequency of the oscillator described by (11) is given by the familiar harmonic formula $\omega = \sqrt{\alpha(q)/M}$ which leads precisely to the dispersion relation eq^n (6) found earlier.

Phase and Group velocities :-

In wave theory, a distinction is made between two kinds of velocities: phase velocity and group velocity. For an arbitrary dispersion relation, phase velocity is given by

$$v_p = \omega/q \quad (12)$$

and group velocity by $v_g = \frac{\partial\omega}{\partial q}$ (13)

The physical distinction between these velocities is that v_p is the velocity of propagation for a pure wave of an exactly specified frequency ω and a wave vector q, while v_g describes the velocity of a wave pulse whose average frequency and wave vector are specified by ω and q. Since, energy and momentum are transmitted, ~~in practice~~, via pulses rather than by pure waves, group velocity is physically the more significant.

We now examine the behaviour of v_g for the discrete lattice. In the long wavelength limit, in which $\omega = v_s q$, v_s is equal to v_p and both are equal to the velocity of sound v_s. In this limit the lattice behaves as a continum, and no dispersion takesplace. But, as q increases, in dispersion curve, that v_g, being the slope of the dispersion curve, decreases steadily and reaches a value $v_g = 0$ at the point $q = \pi/a$. The reason for this decredse is that, as q increases, the scattering of the wave by discrete atoms becomes more pronounced.

A particularly interesting situation arises at $q = \pi/a$ where the group velocity v_g is found to vanish.

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