The Independent particle model (IPM) :->

The basic idea of this model comes from the fact that the nucleons interact with each other through strong attractive short range interactions and scatter each other repeatedly. This multiple scattering is assumed to give rise to an average potential. The average potential so created is such that it exhausts all of the internucleon force strength. This means that the particles, after giving rise to the average potential, move freely, independent of each other in this average potential. The physical basis for this assumption is that, because of the Pauli principle, no two nucleons can occupy the same state, and this inhibits scattering of nucleons in the occupied states. In mathematical terms, this means that the P.E $\sum_{i<j} V_{ij}$ where, V_{ij} is the interaction betⁿ particles i and j, can be written as $\sum_i U_{i-j}$.

[Acco. to the assumption of the IPM, "each nucleon moves independently of other nucleons in an average potential produced by rest of the nucleons."

The average potential is assumed to be a harmonic oscillator potential.

Then, the motion of a particle moving in a 3-D.al oscillator is described by the Hamiltonian.

$$H = \frac{p^2}{2m} + \frac{1}{2} m (x^2\omega_x^2 + y^2\omega_y^2 + z^2\omega_z^2) \quad \text{--- (1)}$$

where, ω_x, ω_y & ω_z are the angular frequencies in x, y and z directions respectively.

For an isotropic oscillator ②

$$\omega_x = \omega_y = \omega_z = \omega$$

then we have

$$H = \frac{p^2}{2m} + \frac{1}{2} m r^2 \omega^2 \quad \text{—— (2)}$$

The allowed energies of the particle are given by ~~the~~ ~~exp~~

$$E_n = \left(2n + l + \frac{3}{2}\right)\hbar\omega, \quad \text{—— (3)}$$

where, n and l are the radial and orbital quantum no.s respectively, $n = 0, 1, 2, 3 \ldots$ and $l = 0, 1, 2, \ldots n-1$.

The ~~energy~~ ~~E_{nl}~~ eigenfunctions corresponding to (energy shell) E_{nl} are given by,

$$|nlm\rangle \equiv \psi_{nlm}(r, \theta, \phi) = R_{nl}(r) Y_{lm}(\theta, \phi), \quad \text{—— (4)}$$

Where $Y_{lm}(\theta, \phi)$ are the normalised spherical harmonics, and the radial functions $R_{nl}(r)$ are given by

$$R_{nl}(r) = \left[\frac{2n}{\Gamma(n + l + 3/2)}\right]^{1/2} r^l \exp(-r^2/2) L_n^{l+1/2}(r^2) \quad \text{—— (5)}$$

Here, $\Gamma(n + l + 3/2)$ is a gamma func. and $L_n^{l+1/2}(r^2)$ are the Laguerre polynomials normalised in such a way that

$$\int_0^\infty [R_{nl}(r)]^2 r^2 dr = 1.$$

The quntum no. m takes on all integral values from $+l$ to $-l$ such that the state $|nl\rangle$ is $(2l+1)$-fold degenerate w.r.t m.

In eq^n (3) for $|n=0, l=0\rangle$ the energy eigen value is $E_{00} = (3/2)\hbar\omega$. This is Known as the zero pt. energy of the oscillator. The state corresponding to the energy of $(5/2)\hbar\omega$ is $|n=0, l=1\rangle$. Howerver for $(7/2)\hbar\omega$ energy there is two allowed states, namely $|n=0, l=2\rangle$ & $|n=1, l=0\rangle$. Thus, these two states are degenerate and have the same energy $(7/2)\hbar\omega$ and so on. Due to m-degeneracy each l level is $(2l+1)$-fold degenerate.

In case of nucleons i.e fermions, having spin of $\pm\frac{\hbar}{2}$ (Z-component), ~~be~~ degeneracy becomes $2(2l+1)$. Thus, $l=0$ is called a s-state, $l=1$ p-state, $l=2$ a d-state and so on. Thus, s-state can accomodate maximum 2 neutrons (and 2 protons), p-state can accommodate a maximum no. of 6 neutrons (and 6 protons), etc.

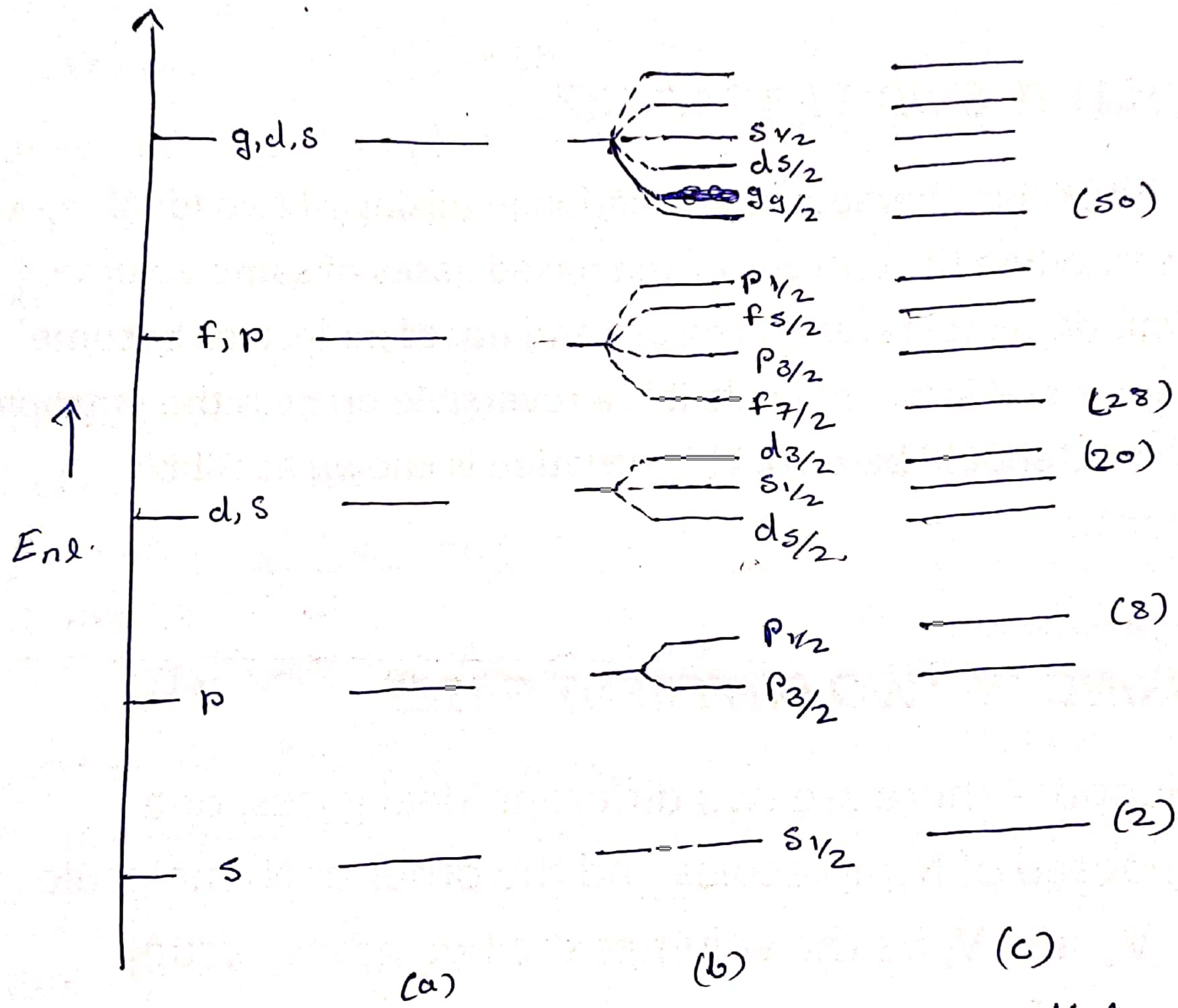

Fig. 1. Single-particle energy levels of a particle moving in a harmonic oscillator well.
(a) denotes unperturbed level scheme,
(b) denotes the effect of spin-orbit coupling,
(c) denotes the resulting level sequence.

Associated magnic no. with filling up of energy shells is shown in fig. 1. Fig. 1 shows ~~that~~ the magic no.s are 2, 8, 20, 40, ~~70~~, etc. However, only the first three of these no.s agree with experiments.

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In order to obtain a satisfactory agreement with experiments, Mrs. Mayer introduced a spin-orbit interaction $-v(r)\vec{l}\cdot\vec{s}$ as a perturbing term to Eq^n (2).
Thus, the single particle Hamiltonian becomes.

$$H = \frac{p^2}{2m} + \frac{1}{2} m r^2 \omega^2 + H' \quad \text{—— (6)}$$

where $\quad H' = -v(r)\vec{l}\cdot\vec{s} \quad$ —— (7)

and $v(r)$ is positive. and the sign of $\vec{l}\cdot\vec{s}$ term in eq^n (7) is of opposite sign, because of the nature of the force.

The eigen functions corresponding to the above Hamiltonian are $|nlm_l\rangle|sm_s\rangle$. The state func. $|nlm_l\rangle$ is given by eq^n (4) and $|sm_s\rangle$ denotes the spin state of the particle with spin $\vec{s}$ and its Z-comp. $S_z = m_s$.

The state of the particle is characterize by quantum no. $\vec{j} = \vec{l}+\vec{s}$ (8) (Known as the total angular momentum) is now denoted as $|nljm\rangle$, where, m is the Z-comp. of $\vec{j}$. This state fun. is related to the $|nlm_l\rangle|sm_s\rangle$ state function by the relation.

$$|nljm\rangle = \sum_{m_l, m_s} \langle l m_l s m_s | jm\rangle |nlm_l\rangle |sm_s\rangle, \quad \text{—— (9)}$$

where, $\langle lm_l s m_s | jm\rangle$ is known as the Clebsch-Gordan Coefficient.

The term H' splits the states with the same unperturbed energy but diff^n j-values.

The expectation value of $\vec{l}\cdot\vec{s}$ is a single particle state with the total angular momentum $\vec{j}$ is

$$\langle \vec{l}\cdot\vec{s}\rangle_j = \frac{1}{2}\left[j(j+1) - l(l+1) - s(s+1)\right]\hbar^2 \quad \text{—— (10)}$$

$\vec{j}$ can take the value $|\vec{j}| = l \pm \frac{1}{2}$

put $\hbar = 1$ for spin or angular momentum

Thus,

$$\langle \vec{l}\cdot\vec{s} \rangle_{j=l+\frac{1}{2}} = \frac{l}{2} \qquad (11)$$

and

$$\langle \vec{l}\cdot\vec{s} \rangle_{j=l-\frac{1}{2}} = -\frac{1}{2}(l+1) \qquad (12)$$

The splitting betn $j = l+\frac{1}{2}$ and $j = l-\frac{1}{2}$ levels due to the spin-orbit coupling term is then

$$\Delta E_{l\cdot s} = \langle H' \rangle_{j=l+\frac{1}{2}} - \langle H' \rangle_{j=l-\frac{1}{2}}$$

$$= \frac{l}{2} \langle v(r) \rangle_{j=l+\frac{1}{2}} + \frac{1}{2}(l+1) \langle v(r) \rangle_{j=l-\frac{1}{2}} \qquad (13)$$

If $\langle v(r) \rangle = c$ (a const.), then this reduces to

$$\Delta E_{l\cdot s} = c\left[\langle \vec{l}\cdot\vec{s} \rangle_{j=l+\frac{1}{2}} - \langle \vec{l}\cdot\vec{s} \rangle_{j=l-\frac{1}{2}}\right]$$

$$= c\,\frac{2l+1}{2} \qquad (14)$$

It is noticed from ~~fie~~ eqn (13) that for an attractive potential i.e. $v(r)$ positive.

(i) The level with a smaller j-value ($j = l-\frac{1}{2}$) lies higher than the one with the larger j-value ($j = l+\frac{1}{2}$)

(ii) Smaller the l-value, smaller is the splitting betn $j = l \pm \frac{1}{2}$ levels, and larger the l-value, larger is the splitting betn these two states. This is shown is Fig. 1.4 b.

It is seen that interpretation for the occurrence of magic no.s is correct. It is obvious that the degeneracy of each level is now $(2j+1)$ instead of $2(2l+1)$. Therefore, 2, 8 and 20 are still the magic no.s (20 filling up the $0d_{3/2}$ state) because of the energy seperations (gaps). The next is the isolated $0f_{7/2}$ level which can accomodate 8 particles, it provides a magic no. 28. Again, $0g_{9/2}$ state comes close in energy to the $1p_{1/2}$ state. ∴, states upto $1p_{1/2}$ cannot provide a magic no.

Instead states upto $0g_{9/2}$ level account for the magic no. 50. ⑥
in agreement with experiment. Thus, it is clear from the ~~above~~ discussion, that the magic no's are accounted ~~for~~ on the basis of energy gaps betⁿ two levels rather than filling up of energy shells. This was importance of $\vec{l}\cdot\vec{s}$ term introduced by Mrs. Maria Mayer.
[Subsequently, another term proportional to $\vec{l}^2$ was also added to eqⁿ 7. In the absence of $\vec{l}\cdot\vec{s}$ term, the degeneracy of various l-levels was splitted in the same energy shell. This is also interpreted as the centrifugal term.

The second assumption of the IPM is that the nucleus tend to pair up in the lowest energy state, i.e they couple to provide a zero spin state, and the magnetic and electric moments of such a stage will also be zero.

For ex – If there are two particles in a $0d_{5/2}$ state then each of these particles will have a $j = 5/2$. The total angular momentum $\vec{J}$ is obtained by the vector addition $\vec{J} = \vec{j}_1 + \vec{j}_2$ and $|j_1 - j_2| \leq J \leq j_1 + j_2$. ∴ a 'paired state' can arise only when $j_1 = j_2$ and $J = j_1 - j_2 = 0$. Hence, if one has an even-Z even-N nucleus, then the ground state spin of such a nucleus will be zero, as all the protons as well as the neutrons will pair up. For an odd A nucleus all the particles, except the last odd particle will get paired up. ∴, this last odd particle will provide, according to IPM, all the properties of the nucleus.