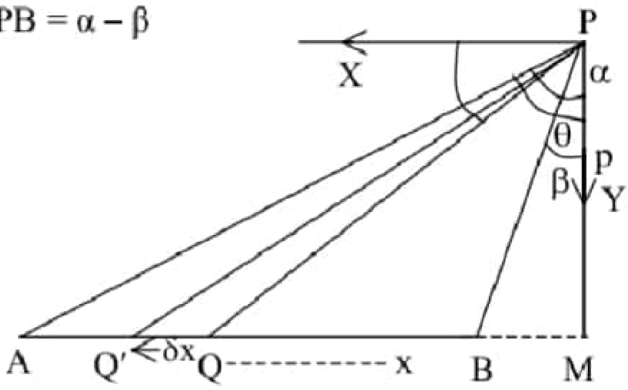
Attraction of a uniform straight rod at an external point:-

AB be a rod

$\angle APB = \alpha - \beta$



Let m be the mass per unit length of a uniform rod AB.

It is required to find the components of attraction of the rod AB at an external point P.

$MP = p$

Consider an element QQ′ of the rod where

$MQ = x$

$QQ' = dx$

$\angle MPQ = \theta$,

In Δ MPQ ,

$$\tan\theta = \frac{MQ}{MP} = \frac{x}{p} \Rightarrow x = p\tan\theta \qquad \ldots(*)$$

$$\cos\theta = \frac{MP}{PQ} = \frac{p}{pQ}$$

$$\Rightarrow \quad PQ = \frac{p}{\cos\theta} \Rightarrow PQ = p\sec\theta \qquad \ldots(**)$$

Mass of element QQ′ of rod = m dx

$$= mp\sec^2\theta\, d\theta \qquad \ldots(\text{using}*)$$

The attraction at P of the element QQ′ is $= \dfrac{\text{mass}}{(\text{distance})^2} = \dfrac{mp\sec^2 d\theta}{(PQ)^2}$ along PQ

Therefore, Force of attraction at P of the element QQ′ is

$$= \frac{mp \sec^2\theta \, d\theta}{p^2 \sec^2\theta}$$ ...[using(**)]

$$= \frac{m}{p} d\theta \qquad \text{along PQ}$$

...(1)

Let $\angle MPA = \alpha$ and $\angle MPB = \beta$

$$f = \int_\beta^\alpha \frac{m}{p} d\theta$$

let X and Y be the components of attraction of the rod parallel & $\perp_r$ to rod, then

$$X = \int_\beta^\alpha \frac{m}{p} \sin\theta \, d\theta$$

& $$Y = \int_\beta^\alpha \frac{m}{p} \cos\theta \, d\theta$$

$$X = \frac{m}{p}[-\cos\theta]_\beta^\alpha = \frac{m}{p}[\cos\beta - \cos\alpha]$$

$$= \frac{m}{p}\left[2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha-\beta}{2}\right]$$ ...(2)

and $$Y = \frac{m}{p}[\sin\theta]_\beta^\alpha = \frac{m}{p}[\sin\alpha - \sin\beta]$$

$$= \frac{m}{p}\left[2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right]$$ ...(3)

Resultant force of Attraction R is given by

$$R = \sqrt{X^2 + Y^2}$$

$$R = \frac{2m}{p}\sin\frac{\alpha-\beta}{2}$$ [...using 2&3]

$$= \frac{2m}{p}\sin\frac{\angle APB}{2}$$

Resultant R makes angle $\tan^{-1}\frac{X}{Y}$

or $\frac{1}{2}(\alpha+\beta)$ with PM $\left[\Theta \tan^{-1}\left(\frac{X}{Y}\right)=\left[\tan^{-1}\left(\tan\frac{\alpha+\beta}{2}\right)\right]\right]$

i.e. it acts along bisector of angle $\angle APB$.

Also $X=\frac{m}{PB}-\frac{m}{PA}$ $\left[\Theta \cos\beta=\frac{p}{PB}, \cos\alpha=\frac{p}{PA} \text{ \& using}(2)\right]$

Cor :- If the rod is infinitely long, the angle APB is two right angles & Resultant attraction $=\frac{2m}{p}\perp_r$ to the rod.

Potential of uniform rod :-

By definition, the potential at P is given by

$$V=\int\frac{m}{PQ}dx$$

$$V=\int_{\beta}^{\alpha}\frac{mp\sec^2\theta}{p\sec\theta}d\theta$$

$$=\int_{\beta}^{\alpha}m\sec\theta\, d\theta$$

$$=m\left[\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]_{\beta}^{\alpha}$$

$$=m\left[\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)-\log\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right)\right]$$

$$=m\log\left[\frac{\tan\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{\beta}{2}+\frac{\pi}{4}\right)}\right]$$

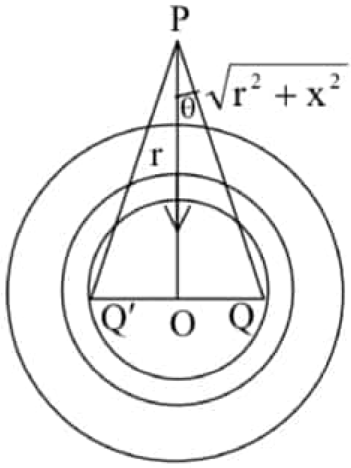
Potential at a point P on the axis of a Uniform circular disc or plate:-

We consider a uniform circular disc of radius 'a' & P is a pt. on the axis of disc & the pt. P is at a distance r from the centre 0, i.e., OP = r, OQ = x, PQ = $\sqrt{r^2+x^2}$ let us divide the disc into a number of concentric rings & let one such ring has radius 'x' & width dx

Then, Mass of ring is $=2\Pi x\, dx\rho$

where ρ density of material of disc $\rho = \text{mass/Area}$

Therefore, Potential at P due to this ring is given by ,$dv = \frac{2\pi x\, dx \rho}{\sqrt{r^2 + x^2}}$



OP = r

Hence, the potential at P due to the whole disc is given by

$$V = 2\Pi\rho \int_0^a \frac{x\, dx}{\sqrt{x^2 + r^2}}$$

$$V = \frac{2\pi\rho}{2} \int_0^a 2x\left(x^2 + r^2\right)^{-\frac{1}{2}} dx$$

$$V = 2\Pi\rho \left[\sqrt{a^2 + r^2} - r\right]$$

Let Mass of disc = M

$$= \Pi a^2 \rho$$

$$\Rightarrow \Pi\rho = \frac{M}{a^2}$$

Then $V = \frac{2m}{a^2}\left[\sqrt{a^2 + r^2} - r\right]$ is requiredpotential at any pt P which lies on the axis of disc.

Attraction at any point on the axis of Uniform circular disc :-

Here radius of disc = a

$OP = r,\qquad PQ = \sqrt{x^2 + r^2}$

$OQ = x$

We consider two element of masses d_m at the two opposite position Q and Q′as shown. Now element d_m at Q causes attraction on unit mass at P in the direction PQ. Similarly other mass d_m at Q′ causes attraction on same unit mass at P in the direction P Q′ and the force of attraction is same in magnitude.

These two attraction forces when resolved into two direction one along the axes PO and other at right angle PO. Components along PO are additive and component along perpendicular to PO canceling each other

Mass of ring = $2\Pi x \, dx \, \rho$

Attraction at P due to ring along PO is given by

$$d\vec{f} = \frac{\left(\sum dm\right)\cos\theta}{(PQ)^2}$$

$$d\vec{f} = \frac{\cos Q.2\pi x\, dx\rho}{(PQ)^2} = \frac{r.2\pi x dx\, \rho}{(PQ)^3} \quad \text{along PO} \qquad [\Theta \cos\theta = \frac{r}{PQ} \text{in} \Delta OPQ\,]$$

$$= \frac{2\pi\rho.rx\, dx}{\left(r^2 + x^2\right)^{\frac{3}{2}}}$$

Therefore, the resultant attraction at P due to the whole disc along PO is given by

$$\vec{f} = \pi\rho r \int_0^a (2x)\left(r^2 + x^2\right)^{-\frac{3}{2}} dx$$

$$= \pi\rho r \left[-2\left(x^2 + r^2\right)^{-\frac{1}{2}}\right]_0^a$$

$$= 2\pi\rho r \left[\frac{1}{r} - \frac{1}{\sqrt{a^2 + r^2}}\right] \text{along PO}$$

Let M = mass of disc of radius a

$=\Pi a^2$

$$\Pi\,\rho = \frac{M}{a^2}\rho$$

$$\vec{f} = \frac{2M}{a^2}\left[1 - \frac{r}{\sqrt{a^2 + r^2}}\right]$$

$$= \frac{2M}{a^2}[1-\cos\alpha]$$

Where α is the angle which any radius of disc subtends at P

Particular cases :-

1. If radius of disc becomes infinite, then $\alpha = \frac{\pi}{2}$

and

$$\overset{\rho}{f} = \frac{2M}{a^2}\left[1-\cos\frac{\pi}{2}\right]$$

$$= \frac{2M}{a^2} = \text{constant [here, it is independent of position of P]}$$

2. When P is at a very large distance from the disc, then $\alpha \to 0$

Therefore, $\overset{\rho}{f} = \frac{2M}{a^2}(1-\cos 0)$

$$= 0$$

Potential of a thin spherical shell :-

We consider a thin spherical shell of radius 'a' & surface density 'ρ' let P be a point at a distance 'r' from the center O of the shell. We consider a slice BB′C′C in the form of ring with two planes close to each other and perpendicular to OP.

Area of ring (slice) BB′C′C

$$=2\Pi BD \times BB'$$

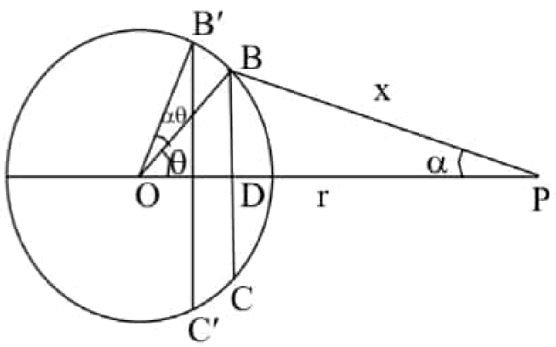
where Radius of ring, $BD = a\sin\theta$

width of ring, $BB' = a\,d\theta$

Therefor,Mass of slice (ring) is

$$= 2\Pi\, a\sin\theta\, a d\theta\, \rho$$

$$= 2\Pi a^2\, \rho \sin\theta\, d\theta$$



Hence, Potential at P due to slice (ring) is

$$dV = \frac{2\pi a^2 \rho \sin\theta d\theta}{x} \qquad \dots(1)$$

Now, from ΔBOP,

$$BP^2 = OP^2 + OB^2 - 2OP.OB\cos\theta$$

$$x^2 = r^2 + a^2 - 2ar\cos\theta$$

differentiating

$$2x\,dx = 2ar\sin\theta\,d\theta$$

$$\frac{x}{ar}dx = \sin\theta d\theta$$

Putting in (1), we get $dV = \frac{2\pi a^2 \rho x\,dx}{x.ar}$

$$= \frac{2\pi a\rho dx}{r} \qquad (2)$$

Therefore, Potential for the whole spherical shell is obtained by integrating equation (2), we have

$$V = \int \frac{2\pi a\rho}{r} dx$$

$$= \frac{2\pi a\rho}{r}\int dx$$

Now, we consider the following cases :-

Case(i) The point P is outside the shell. In this case, the limit of integration extends from x = (r – a) to x = (r + a)

Hence $$V = \frac{2\pi a\rho}{r}\int_{r=a}^{r+a} dx$$

$$V = \frac{4\pi a^2 \rho}{r}$$

Here, Mass of spherical shall = $4\Pi a^2 \rho$

Then $V = \frac{M}{r}$

Case(ii) When P is on the spherical shell, then limits are from x = 0 to x = 2a (here r = a)

Then $V = \frac{2\pi a\rho}{a}\int_0^{2a} dx$

$$= \frac{4\pi a^2\rho}{a} = \frac{M}{a}$$

Case(iii) When P is inside the spherical shell, limit are from $x = (a - r)$ to $(a + r)$

$$V = 4\Pi a\,\rho = \frac{M}{a}$$

Attraction of a spherical shell

Let us consider a slice BB´C´C at point P , the attraction due to this slice is

$$d\vec{f} = \frac{2\pi a^2\rho\sin\theta\, d\theta}{x^2} \text{ along PB}$$

The resultant attraction directed along PO is given by

$$d\vec{f} = \frac{2\pi a^2\rho\sin\theta\, d\theta}{x^2}\cos\alpha$$

We know that $\sin\theta\, d\theta = \frac{xdx}{ar}$

In ΔBDP, $\cos\alpha = \frac{PD}{PB} = \frac{r - a\cos\theta}{x}$.

$$d\vec{f} = \frac{2\pi a^2\rho xdx}{ar.x^2}\left(\frac{r - a\cos\theta}{x}\right)$$

We know that

$$x^2 = a^2 + r^2 - 2ar\cos\theta$$

$$x^2 - a^2 + r^2 = 2r^2 - 2ar\cos\theta$$

$$\frac{x^2 - a^2 + r^2}{2r} = r - a\cos\theta$$

Then, $d\vec{f} = \frac{2\pi a^2\rho xdx}{ar.x^2}\frac{(x^2 - a^2 + r^2)}{2r, x}$

$$= \frac{\pi a\rho}{r^2}\left(\frac{x^2 - a^2 + r^2}{x^2}\right)dx$$

$$= \frac{a\pi\rho}{r^2}\left(1+\frac{r^2+a^2}{x^2}\right)dx$$

Hence the attraction for the whole spherical shell is obtained by integration

Therefore, $\overset{\rho}{f} = \frac{\pi a\rho}{r^2}\int\left[1+\frac{r^2-a^2}{x^2}\right]dx$

Now we consider the following cases depending upon the position of P

Case(i) When point P is inside the shell, the limits of integration are x = (r – a) to (r + a)

$$\vec{f} = \frac{\pi a\rho}{r^2}\int_{r-a}^{r+a}\left(1+\frac{r^2-a^2}{x^2}\right)dx$$

$$\vec{f} = \frac{\pi a\rho}{r^2}\left[x+\left(r^2-a^2\right)\left(\frac{-1}{x}\right)\right]_{r-a}^{r+a}$$

$$= \frac{4\pi a^2\rho}{r^2} = \frac{M}{r^2}$$

Case(ii) When pt. P is on the shell, the limit of integration are x = 0 to 2a

$$\overset{\rho}{f} = \frac{\pi a\rho}{r^2}\int_{0}^{2a}\left(1+\frac{r^2-a^2}{x^2}\right)dx$$

Here integration is not possible (due to second term is becoming indeterminant), because when P is on the shell, then

$$r = a;\ x = 0$$

Hence to evaluate the integral, we consider that pt. P is situated not on the surface but very near to the surface

Let $r = a + \delta$, where δ is very small

Then attraction is $\overset{\rho}{f} = \frac{a\pi\rho}{r^2}\left[\int_{\delta}^{2a+\delta}dx + \int_{\delta}^{2a+\delta}\left\{\frac{(a+\delta)^2-a^2}{x^2}\right\}dx\right]$

$$= \frac{\pi a\rho}{r^2}\left[2a + \int_{\delta}^{2a+\delta}\frac{2a\delta}{x^2}dx\right]$$

$$= \frac{\pi a \rho}{r^2}\left[2a + 2a\delta\left(\frac{-1}{x}\right)_{\delta}^{2a+\delta}\right]$$

$$= \frac{\pi a \rho}{r^2}\left[2a - \frac{2a\delta}{2a+\delta} + \frac{2a\delta}{\delta}\right]$$

$$= \frac{2\pi a^2 \delta}{r^2}\left[2 - \frac{\delta}{2a+\delta}\right] \text{ as } \delta \to o, \text{ then } r = a$$

$$= \frac{4\pi a^2 \rho}{a^2} = \frac{M}{a^2}$$

Case (iii) Poin. P is inside the shell, limits are $x = a - r$ to $a + r$

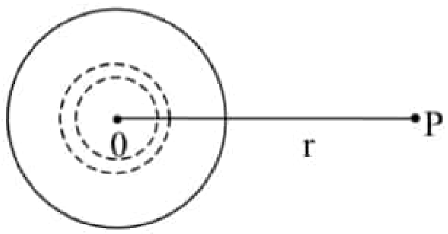
$$\overset{\rho}{f} = \frac{\pi a \rho}{r^2}\int_{a-r}^{a+r}\left[1 + \frac{r^2 + a^2}{x^2}\right]dx$$

$$= \frac{\pi a \rho}{r^2}\left[x - \left(r^2 - a^2\right)\left(\frac{1}{x}\right)\right]_{a-r}^{a+r} = 0$$

So, there is no resultant attraction inside the shell.

Potential of a Uniform solid sphere:- A uniform solid sphere may be supposed to be made up of a number of thin uniform concentric spherical shells. The masses of spherical shells may be supposed to be concentric at centre O.

Case I :- At an external point



Therefore the potential due to all such shells at an external point P is given by

$$V = \frac{m_1}{r} + \frac{m_2}{r} + \ldots$$

where m_1, m_2 … etc are the masses of shells.

$$V = \frac{1}{r}(m_1 + m_2 + \ldots) = \frac{M}{r}$$

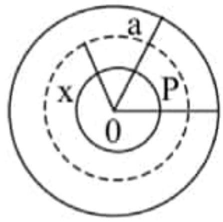
where M is the mass of solid sphere.

Case II:- The point P is on the sphere.

In case I, put r = a

$$V = \frac{M}{a}, \text{ where } a = \text{radius of sphere}$$

Case III:- At an internal point. Here point P is considered to be external to solid sphere of radius r & internal to the shell of internal radius r, external radius = a.



Let V_1 = potential due to solid sphere of radius r

V_2 = potential due to thick shell ofinternal radius r and external radius a

Then $$V_1 = \frac{\text{mass of sphere of radius r}}{r}$$

$$= \frac{4}{3}\frac{\pi r^3 \rho}{r} = \frac{4}{3}\pi r^2 \rho$$

To calculate V_2

We consider a thin concentration shell of radius 'x' & thicknes dx. The potential at P due to thin spherical shell under consideration is given by

$$\frac{4\pi x^2 dx\rho}{x} = 4\pi x \, dx \, \rho$$

Hence for the thick shell of radius r & a, the potential is given by

$$V_2 = 4\pi\rho \int_r^a x \, dx$$

$$= 4\pi\rho\left(\frac{a^2 - r^2}{2}\right) = 2\pi\rho(a^2 - r^2)$$

Therefore, the potential at P due to given solid sphere.

$$V = V_1 + V_2 = \frac{2}{3}\pi\rho(3a^2 - r^2)$$

where M = Mass of given solid sphere $= \frac{4}{3}\pi a^3 \rho$

$$\Rightarrow \quad \pi\rho = \frac{3M}{4\pi a^3}$$

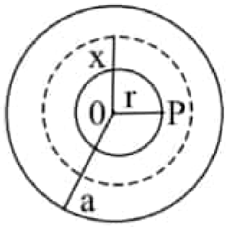
Hence $V = \frac{2}{3} \cdot \frac{3M}{4\pi a^3}(3a^2 - r^2) = \frac{M}{2a^3}(3a^2 - r^2)$

Attraction for a uniform solid sphere

Case I : At an external point

$$\vec{F} = \frac{m_1}{r^2} + \frac{m_2}{r^2} + \ldots.$$

$$\vec{F} = \frac{M}{r^2}, \quad M = m_1 + m_2 + \ldots.$$



M = Mass of sphere and m_1 , m_2…. Are masses of concentric spherical shells

Case II: At a point on the sphere,

Here we put r = a in above result

We get $\vec{F} = \frac{M}{a^2}$

Case III: At a point inside the sphere.

The point P is external to the solid sphere of radius r and it is internal to thick spherical shell of radii r and a.

And we know that attraction (forces of attraction) at an internal point in case of spherical shell is zero. Hence the resultant attraction at P is only due to solid sphere of radius r and is given by

$$\vec{F} = \frac{\text{mass of sphere of radius r}}{r^2}$$

$$= \frac{4}{3}\frac{\pi r^3 \rho}{r^2} = \frac{4}{3}\pi r \rho$$

If $M = \frac{4}{3}\pi a^3 \rho \;\Rightarrow\; \pi\rho = \frac{3M}{4a^3}$

Then $\vec{F} = \frac{Mr}{a^3}$