Lagrange's Brackets

Lagrange's bracket of (u, v) w.r.t. the basis (q_j, p_j) is defined as

$$\{u, v\}_{q,p} \text{ or } (u, v)_{q,p} = \sum_j \left[\frac{\partial q_j}{\partial u}\cdot\frac{\partial p_j}{\partial v} - \frac{\partial p_j}{\partial u}\frac{\partial q_j}{\partial v}\right]$$

Properties: (1) (u, v) = −(v, u)

(2) $(q_i, q_j) = 0$

(3) $(p_i, p_j) = 0$

(4) $(q_i, p_j) = \delta_{ij}$

$$(u, v) = \sum_j \left(\frac{\partial q_j}{\partial u}\frac{\partial p_j}{\partial v} - \frac{\partial p_j}{\partial u}\frac{\partial q_j}{\partial v}\right) = -\sum_j \left(\frac{\partial p_j}{\partial u}\frac{\partial q_j}{\partial v} - \frac{\partial q_j}{\partial u}\frac{\partial p_j}{\partial v}\right) = -(v, u)$$

(2) $(q_i, q_j) = \sum_k \left(\frac{\partial q_k}{\partial q_i}\cdot\frac{\partial p_{k'}}{\partial q_j} - \frac{\partial q_{k'}}{\partial q_j}\frac{\partial p_k}{\partial q_i}\right)$ [Since q's and p's are independent

$= 0.$ $\Rightarrow \frac{\partial p_{k'}}{\partial q_j} = o$ and $\frac{\partial p_k}{\partial q_i}$ 0]

(3) Similarly we can prove that

$$\{p_i, p_j\} = 0$$

(4) $\{q_i, p_j\} = \sum_k \left(\frac{\partial q_k}{\partial q_i}\cdot\frac{\partial p_k}{\partial p_j} - \frac{\partial q_k}{\partial p_j}\frac{\partial p_k}{\partial q_i}\right) = \sum_k \frac{\partial q_k}{\partial q_i}\cdot\frac{\partial p_k}{\partial p_j} = \sum_k \delta_{ki}\delta_{kj} = \delta_{ij}$

Invariance of Poisson's Bracket under Canonical transformation:-

Poisson's bracket is

$$(u, v)_{q,p} = \sum_j \left(\frac{\partial u}{\partial q_j}\frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q_j}\right)$$

The transformation of co- ordinates in a 2n – dimensional phase space is called canonical if the transformation carries any Hamiltonian into a new hamiltonian system

To show :- $[F,G]_{q,p} = [F, G]_{Q,P}$

Poisson's brackets is

$$[F, G]_{q,p} = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)$$

If q, p are functions of Q & P then q = q (Q, P) & p = p (Q, P) and F & G will also function of (q, p), we have, $G = G(Q_k, P_k)$, we have

$$[F, G]_{q,p} = \sum_i \left\{ \begin{aligned} &\frac{\partial F}{\partial q_i}\left[\frac{\partial G}{\partial Q_k}\cdot\frac{\partial Q_k}{\partial P_i} + \frac{\partial G}{\partial P_k}\cdot\frac{\partial p_k}{\partial P_i}\right] \\ &-\frac{\partial F}{\partial p_i}\left[\frac{\partial G}{\partial Q_k}\cdot\frac{\partial Q_k}{\partial q_i} + \frac{\partial G}{\partial P_k}\cdot\frac{\partial P_k}{\partial q_i}\right] \end{aligned} \right\}$$

$$= \sum_{i,k} \left\{ \begin{aligned} &\frac{\partial G}{\partial Q_k}\left[\frac{\partial F}{\partial q_i}\cdot\frac{\partial Q_k}{\partial p_i} - \frac{\partial F}{\partial p_i}\cdot\frac{\partial Q_k}{\partial q_i}\right] \\ &+\frac{\partial G}{\partial P_k}\left[\frac{\partial F}{\partial q_i}\cdot\frac{\partial P_k}{\partial p_i} - \frac{\partial F}{\partial p_i}\cdot\frac{\partial P_k}{\partial q_i}\right] \end{aligned} \right\}$$

$$= \sum_{i,k} \left\{ \frac{\partial G}{\partial Q_k}\left[F, Q_q\right]_{q,p} + \frac{\partial G}{\partial P_k}\left[F, P_k\right]_{q,p} \right\} \qquad \ldots(1)$$

To find $[F, Q_k]_{q,p}$ & $[F, P_k]_{q,p}$

Replacing F by Q_i in (1)

$$[Q_i, G]_{q,p} = \sum_{i,k} \frac{\partial G}{\partial Q_k}[Q_i, Q_k]_{q,p} + \frac{\partial G}{\partial P_R}[Q_i, P_k]_{q,p}$$

$$= 0 + \sum_k \frac{\partial G}{\partial P_k}\delta_{ik}$$

$$[Q_i, G]_{q,p} = \frac{\partial G}{\partial P_i}$$

$$\Rightarrow \quad [G, Q_i]_{q,p} = -\frac{\partial G}{\partial P_i}$$

and $\quad [F, Q_k]_{q,p} = -\dfrac{\partial F}{\partial P_R} \qquad \ldots(2)$

Replacing F by P_i in (1)

$$[P_i, G] = -\frac{\partial G}{\partial \varphi_i} \quad \Rightarrow \quad [G, P_i] = \frac{\partial G}{\partial q_i}$$

and $\quad [F, P_k] = \dfrac{\partial F}{\partial q_k} \qquad \ldots(3)$

Put these values from (2) and (3) in (1), we get:-

$$[F, G]_{q,p} = \sum_{i,k}\left(-\frac{\partial G}{\partial \phi_K}\frac{\partial F}{\partial P_K} + \frac{\partial G}{\partial P_K}\cdot\frac{\partial F}{\partial \phi_K}\right).$$

$= [F, G]_{Q,P}$

Poincare integral Invariant:-

Under Canonical transformation, the integral

$$J = \iint_S \sum dq_i dp_i \qquad \ldots(1)$$

Where S is any 2 – D (surface) phase space remains Invariant

Proof :- The position of a point on any 2– D surface is specified completely by two parameters, e.g. u, v

Then
$$\left.\begin{aligned} q_i &= q_i(u, v) \\ p_i &= p_i(u, v) \end{aligned}\right\} \qquad \ldots(2)$$

In order to transform integral (1) into new variables (u, v), we take the relation

$$dq_i\, dp_i = \frac{\partial(q_i, p_i)}{\partial(u, v)} du\, dv \qquad \ldots(3)$$

where $\frac{\partial(q_i, p_i)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial p_i}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial p_i}{\partial v} \end{vmatrix}$ as the Jacobian.

Let Canonical transformation be

$$Q_k = Q_k(q, p, t),\ P_k = P_k(q, p, t) \qquad \ldots(4)$$

then
$$dQ_k\, dP_k = \frac{\partial(\varphi_K P_K)}{\partial(u, v)} du\, dv \qquad \ldots(5)$$

if J is invariant under canonical transformation (4), then we can write

$$\iint_S \sum_i dq_i dp_i = \iint_S \sum_K dQ_K dP_K$$

or
$$\iint_S \sum_i \frac{\partial(q_i p_i)}{\partial(u, v)} du\, dv = \iint_S \sum_K \frac{\partial(Q_K, P_K)}{\partial(u, v)} dudv$$

Because the surface S is arbitrary the expressions are equal only if the integrands are identicals,

i.e.,

$$\sum_i \frac{\partial(q_i,p_i)}{\partial(u,v)} = \sum_K \frac{\partial(Q_K,P_K)}{\partial(u,v)}$$

...(6)

In order to prove it, we would transform(q,p) basis to (Q, P) basesthrough the generating function $F_2(q, p, t)$, With this form of generating function, we have

$$p_i = \frac{\partial F_2}{\partial q_i} \ \& \ Q_K = \frac{\partial F_2}{\partial P_k}$$

$$\frac{\partial p_i}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\partial F_2}{\partial q_i}\right) = \sum_K \left(\frac{\partial^2 F_2}{\partial q_i . \partial q_k}\cdot\frac{\partial q_k}{\partial u} + \frac{\partial^2 F_2}{\partial q_i . \partial p_k}\frac{\partial P_k}{\partial u}\right)$$

and $\frac{\partial p_i}{\partial v} = \frac{\partial}{\partial v}\left(\frac{\partial F_2}{\partial q_i}\right) = \sum_K \left(\frac{\partial^2 F_2}{\partial q_i . \partial q_k}\cdot\frac{\partial q_k}{\partial v} + \frac{\partial^2 F_2}{\partial q_i . \partial p_k}\frac{\partial P_k}{\partial v}\right)$

Now, $\sum_i \frac{\partial(q_i,p_i)}{\partial(u,v)} = \sum_i \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial p_i}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial p_i}{\partial v} \end{vmatrix}$

$$= \sum_{i,k} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial^2 F_2}{\partial q_i \partial q_k}\frac{\partial q_k}{\partial u} + \frac{\partial^2 F_2}{\partial q_i \partial P_k}\frac{\partial P_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial^2 F}{\partial q_i \partial q_k}\frac{\partial q_k}{\partial v} + \frac{\partial^2 F_2}{\partial q_i \partial P_k}\frac{\partial P_k}{\partial v} \end{vmatrix}$$

$$= \sum_{i,k} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial^2 F_2}{\partial q_i \partial q_k}\frac{\partial q_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial^2 F_2}{\partial q_i \partial q_k}\frac{\partial q_k}{\partial v} \end{vmatrix} + \sum_{i,k} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial^2 F_2}{\partial q_i \partial P_k}\frac{\partial P_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial^2 F_2}{\partial q_i \partial P_k}\frac{\partial P_k}{\partial v} \end{vmatrix}$$

L. H. S of (6) = $\sum_{i,k} \frac{\partial^2 F}{\partial q_i \partial q_i} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial q_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial q_k}{\partial v} \end{vmatrix}$

$$+ \sum_{i,k} \frac{\partial^2 F}{\partial q_i \partial P_k} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial P_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial P_k}{\partial v} \end{vmatrix} \qquad ...(7)$$

We see that first term on R.H.S. is antisymmetric expression under interchange of i and k, its value will be zero, i.e.,

$$\sum_{i,k}\frac{\partial^2 F}{\partial q_i \partial q_k}\begin{vmatrix}\frac{\partial q_i}{\partial u} & \frac{\partial q_k}{\partial u}\\ \frac{\partial q_i}{\partial v} & \frac{\partial q_k}{\partial v}\end{vmatrix} = \sum_{k,i}\frac{\partial^2 F}{\partial q_k \partial q_i}\begin{vmatrix}\frac{\partial q_k}{\partial u} & \frac{\partial q_i}{\partial u}\\ \frac{\partial q_k}{\partial v} & \frac{\partial q_i}{\partial v}\end{vmatrix}$$

$$= -\sum_{k,i}\frac{\partial^2 F}{\partial q_k \partial q_i}\begin{vmatrix}\frac{\partial q_i}{\partial u} & \frac{\partial q_k}{\partial u}\\ \frac{\partial q_i}{\partial v} & \frac{\partial q_k}{\partial v}\end{vmatrix}$$

or $$\sum_{i,k}\frac{\partial^2 F}{\partial q_i \partial q_k}\begin{vmatrix}\frac{\partial q_i}{\partial u} & \frac{\partial q_k}{\partial u}\\ \frac{\partial q_i}{\partial v} & \frac{\partial q_k}{\partial v}\end{vmatrix} = 0 \qquad \ldots(8)$$

Similarly replacing q by P we have from (8)

$$\sum_{i,k}\frac{\partial^2 F}{\partial P_i \partial P_k}\begin{vmatrix}\frac{\partial P_i}{\partial u} & \frac{\partial P_k}{\partial u}\\ \frac{\partial P_i}{\partial v} & \frac{\partial P_k}{\partial v}\end{vmatrix} = 0$$

Now equation (7) can be written as

$$\sum_{i}\frac{\partial(q_i p_i)}{\partial(u,v)} = \sum_{i,k}\frac{\partial^2 F}{\partial P_i \partial P_k}\begin{vmatrix}\frac{\partial P_i}{\partial u} & \frac{\partial P_k}{\partial u}\\ \frac{\partial P_i}{\partial v} & \frac{\partial P_k}{\partial v}\end{vmatrix}$$

$$+\sum_{i,k}\frac{\partial^2 F}{\partial q_i \partial P_k}\begin{vmatrix}\frac{\partial q_i}{\partial u} & \frac{\partial P_k}{\partial u}\\ \frac{\partial q_i}{\partial v} & \frac{\partial P_k}{\partial v}\end{vmatrix}$$

$$= \sum_{i,k}\begin{vmatrix}\frac{\partial^2 F_2}{\partial P_k \partial P_i}\frac{\partial P_i}{\partial u} + \frac{\partial^2 F_2}{\partial P_k \partial q_i}\frac{\partial q_i}{\partial u} & \frac{\partial P_k}{\partial u}\\ \frac{\partial^2 F}{\partial P_k \partial P_i}\frac{\partial P_i}{\partial v} + \frac{\partial^2 F_2}{\partial P_k \partial q_i}\frac{\partial q_i}{\partial v} & \frac{\partial P_k}{\partial v}\end{vmatrix}$$

$$= \sum_{k}\begin{vmatrix}\frac{\partial}{\partial u}\left(\frac{\partial F_2}{\partial P_k}\right) & \frac{\partial P_k}{\partial u}\\ \frac{\partial}{\partial v}\left(\frac{\partial F_2}{\partial P_k}\right) & \frac{\partial P_k}{\partial v}\end{vmatrix}$$

Put $\frac{\partial F_2}{\partial P_k} = Q_k$

$$= \sum_k \begin{vmatrix} \frac{\partial Q_k}{\partial u} & \frac{\partial P_k}{\partial u} \\ \frac{\partial Q_k}{\partial v} & \frac{\partial P_k}{\partial v} \end{vmatrix} = \sum_k \frac{\partial(Q_k, P_k)}{\partial(u, v)} = \text{R.H.S of (6).}$$

Which proves that integral is invariant under canonical transformation.

Lagrange's bracket is invariant under Canonical transformation:-

The Lagrange's bracket of u & v is defined as

$$\{u, v\}_{q,p} = \sum_i \left(\frac{\partial q_i}{\partial u} \frac{\partial p_i}{\partial v} - \frac{\partial q_i}{\partial v} \frac{\partial p_i}{\partial u} \right)$$

$$= \sum_i \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial q_i}{\partial v} \\ \frac{\partial p_i}{\partial u} & \frac{\partial p_i}{\partial v} \end{vmatrix}$$

Since $\sum_i \frac{\partial(q_i p_i)}{\partial(u, v)}$ is invariant under Canonical transformation.

So Lagrange's bracket is also invariant under canonical transformation