Jacobi's Identity on Poisson Brackets (Poisson's Identity):- If X, Y, Z are function of q & p only, then

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

Proof : $[X, [Y, Z]] + [Y, [Z, X]] = [X, [Y, Z]] - [Y, [X, Z]]$

$$= \left[X, \sum_j \left(\frac{\partial Y}{\partial q_j}\frac{\partial Z}{\partial p_j} - \frac{\partial Y}{\partial p_j}\frac{\partial Z}{\partial q_j}\right)\right]$$

$$\left[Y, \sum_j \left(\frac{\partial X}{\partial q_j}\frac{\partial Z}{\partial p_j} - \frac{\partial X}{\partial p_j}\frac{\partial Z}{\partial q_j}\right)\right] \quad \ldots(1)$$

Let $\sum_j \frac{\partial Y}{\partial q_j}\frac{\partial Z}{\partial p_j} = E, \quad \sum_j \frac{\partial Y}{\partial p_j}\frac{\partial Z}{\partial q_j} = F$

$$\sum_j \frac{\partial X}{\partial q_j}\frac{\partial Z}{\partial p_j} = G, \qquad \sum_j \frac{\partial X}{\partial p_j}\frac{\partial Z}{\partial q_j} = H$$

$\therefore$ (1) $\Rightarrow [X, [Y, Z]] - [Y, [X, Z]]$

$$= [X, E-F] - [Y, G-H]$$

$$= [X, E] - [X, F] - [Y, G] + [Y, H] \quad \ldots(2)$$

Let $E = \sum_j \frac{\partial Y}{\partial q_j}\frac{\partial Z}{\partial p_j} - \left(\sum_j \frac{\partial Y}{\partial q_j}\right)\left(\sum_j \frac{\partial Z}{\partial p_j}\right)$

$\therefore \quad E = E_1 E_2$

Similarly $F = F_1 F_2$, $G = G_1 G_2$, $H = H_1 H_2$

$\therefore$ RHS of (2) becomes

$$[X, E] - [X, F] - [Y, G] + [Y, H] = [X, E_1 E_2] + [Y, H_1 H_2] - [X, F_1 F_2] - [Y, G_1 G_2]$$

$$= [X, E_1] E_2 + [X, E_2] E_1 - [X, F_1] F_2 - [X, F_2] F_1 - [Y, G_1] G_2$$

$$-[Y, G_2] G_1 + [Y, H_1] H_2 + [Y, H_2] H_1$$

$$\therefore \quad \text{RHS of (2) is} = \left[X, \sum_j \left(\frac{\partial Y}{\partial q_j}\frac{\partial Z}{\partial p_j}\right)\right] - \left[X, \sum_j \left(\frac{\partial Y}{\partial p_j}\frac{\partial Z}{\partial q_j}\right)\right]$$

$$-\left[Y, \sum_j \left(\frac{\partial X}{\partial q_j}\frac{\partial Z}{\partial p_j}\right)\right] + \left[Y, \sum_j \left(\frac{\partial X}{\partial p_j}\frac{\partial Z}{\partial q_j}\right)\right]$$

Using properties $[X, E_1 E_2] = [X, E_1] E_2 + [X, E_2] E,$

$$= \left[X, \sum \frac{\partial Y}{\partial q_j}\right]\sum \frac{\partial Z}{\partial p_j} + \left[X, \sum \frac{\partial Z}{\partial p_j}\right]\sum \frac{\partial Y}{\partial q_j}$$

$$-\left[X, \sum \frac{\partial Y}{\partial p_j}\right]\sum \frac{\partial Z}{\partial q_j} - \left[X, \sum \frac{\partial Z}{\partial q_j}\right]\sum \frac{\partial Y}{\partial p_j}$$

$$-\left[Y, \sum \frac{\partial X}{\partial q_j}\right]\sum \frac{\partial Z}{\partial q_j} - \left[Y, \sum \frac{\partial Z}{\partial p_j}\right]\sum \frac{\partial X}{\partial q_j}$$

$$+\left[Y, \sum \frac{\partial X}{\partial p_j}\right]\sum \frac{\partial Z}{\partial q_j} + \left[Y, \sum \frac{\partial Z}{\partial q_j}\right]\sum \frac{\partial X}{\partial p_j}$$

$$= \sum_j \left\{\frac{-\partial Z}{\partial q_j}\left(\left[\frac{\partial X}{\partial p_j}, Y\right] + \left[X, \frac{\partial Y}{\partial p_j}\right]\right) + \frac{\partial Z}{\partial p_j}\left(\left[\frac{\partial X}{\partial q_j}, Y\right]\left[X, \frac{\partial Y}{\partial q_j}\right]\right)\right\}$$

$$+ \sum_j \left\{\frac{\partial Y}{\partial q_j}\left[X, \frac{\partial Z}{\partial p_j}\right] - \frac{\partial Y}{\partial p_j}\left[X, \frac{\partial Z}{\partial q_j}\right] - \frac{\partial X}{\partial q_j}\left[Y, \frac{\partial Z}{\partial p_j}\right] + \frac{\partial X}{\partial p_j}\left[Y, \frac{\partial Z}{\partial q_j}\right]\right\} \quad \ldots(3)$$

Using the identity,

$$\frac{\partial}{\partial t}[X, Y] = \left[\frac{\partial X}{\partial t}, Y\right] + \left[X, \frac{\partial Y}{\partial t}\right]$$

Then, we find that R.H.S. of equation (3) reduces to

$$= \sum_j \left\{ -\frac{\partial Z}{\partial q_j}\frac{\partial}{\partial p_j}[X, Y] + \frac{\partial Z}{\partial p_j}\frac{\partial [X, Y]}{\partial q_j} \right\}$$

+ 0 (All other terms are cancelled)

$$= -\sum_j \left\{ \frac{\partial Z}{\partial q_j}\frac{\partial [X, Y]}{\partial p_j} - \frac{\partial Z}{\partial p_j}\frac{\partial [X, Y]}{\partial q_j} \right\}$$

$$= -[Z, [X, Y]]$$

or $\quad [X, [Y, Z]] + [Y, [Z, X]] = -[Z, [X, Y]]$

$\Rightarrow \quad [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Particular Case

Let Z = H, then

$$[X, [Y, H]] + [Y, [H, X]] + [Y, [X, Y]] = 0$$

Suppose X & Y both are constants of motion, then

$$[X, H] = 0, \quad [Y, H] = 0$$

Then Jacobi's identity gives

$$[H, [X, Y]] = 0$$

$\Rightarrow$ [X, Y] is also a constant of Motion. Hence poisson's Bracket of two constants of Motion is itself a constant of Motion.

Poisson's Theorem

The total time rate of evolution of any dynamical variable F(p, q, t) is given by

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + [F, H]$$

Solution : $$\frac{dF}{dt}(p, q, t) = \frac{\partial F}{\partial t} + \sum_j \left[\frac{\partial F}{\partial q_j}\dot{q}_j + \frac{\partial F}{\partial p_j}\dot{p}_j \right]$$

$$= \frac{\partial F}{\partial t} + \sum_j \left[\frac{\partial F}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial H}{\partial q_j} \right]$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + [F, H]$$

If F is constant of motion, then $\frac{dF}{dt} = 0$.

Then by Poisson's theorem,

$$\frac{\partial F}{\partial t} + [F, H] = 0$$

Further if F does not contain time explicitly, then $\frac{\partial F}{\partial t} = 0$

$\Rightarrow \quad [F, H] = 0$

This is the requirement condition for a dynamical variable to be a constant of motion.

Jacobi-Poisson Theorem :- (or Poisson's Second theorem)

If u and v are any two constants of motion of any given Holonomic dynamical system, then their Poisson bracket [u, v] is also a constant of motion.

Proof:- We consider $\frac{d}{dt}[u,v] = \frac{\partial}{\partial t}[u, v] + [[u, v], H]$ ...(1)

using the following results,

$$\frac{\partial}{\partial t}[u,v] = \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right] \quad ...(2)$$

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0 \quad ...(3)$$

$$\therefore \quad (1) \Rightarrow \frac{d}{dt}[u,v] = \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right] + [[u,v], H] \quad ...(4)$$

Put w = H in (3), we get

$$[H, [u, v]] = -[u, [v, H]] - [v, [H, u]]$$

$$\Rightarrow \quad -[[v, H], u] - [[H, u], v] = [[u, v], H] \quad ...(5)$$

from (4) & (5), we get

$$\frac{d}{dt}[u,v] = \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right] - [[v, H], u] - [[H, u], v]$$

$$= \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right] + [u, (v, H]] + [[v, H], v]$$

$$= \left[\frac{\partial u}{\partial t} + [u, H], v\right] + \left[u, \frac{\partial v}{\partial t} + [v, H]\right]$$

$$\Rightarrow \qquad \frac{d}{dt}[u, v] = \left[\frac{du}{dt}, v\right] + \left[u, \frac{dv}{dt}\right] \qquad \ldots(6)$$

Because $\frac{du}{dt}$ and $\frac{dv}{dt}$ both are zero as u & v were constants of motion.

$\therefore \quad (6) \Rightarrow \quad \frac{d}{dt}[u, v] = 0$

$\Rightarrow$ The Poisson bracket [u, v] is also a constant of motion.