Poisson's Bracket

Let A and B are two arbitrary function of a set of canonical variables (or conjugate variables) $q_1, q_2, \ldots, q_n, p_1, p_2 \ldots p_n$, then Poisson's Bracket of A & B is defined as

$$[A, B]_{q,p} = \sum_j \left(\frac{\partial A}{\partial q_j} \frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial p_j} \frac{\partial B}{\partial q_j} \right)$$

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Properties

I. $[X, Y]_{q,p} = -[Y, X]_{q,p}$

II. $[X, X] = 0$

III. $[X, Y+Z] = [X, Y] + [X, Z]$

IV. $[X, YZ] = Y[X, Z] + Z[X, Y]$

Solution :- I. By definition $[X, Y]_{q,p} = \sum_j \left(\frac{\partial X}{\partial q_j} \frac{\partial Y}{\partial p_j} - \frac{\partial X}{\partial p_j} \frac{\partial Y}{\partial q_j} \right)$

$$[Y, X]_{q,p} = \sum_j \left(\frac{\partial Y}{\partial q_j} \frac{\partial X}{\partial p_j} - \frac{\partial Y}{\partial p_j} \frac{\partial X}{\partial q_j} \right)$$

$$= -\sum_j \left(\frac{\partial X}{\partial q_j} \frac{\partial Y}{\partial p_j} - \frac{\partial X}{\partial p_j} \frac{\partial Y}{\partial q_j} \right)$$

$\Rightarrow \quad [Y, X]_{q,p} = -[X, Y]$

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II. $[X, X]_{q,p} = \sum_j \left(\frac{\partial X}{\partial q_j} \frac{\partial X}{\partial p_j} - \frac{\partial X}{\partial p_j} \frac{\partial X}{\partial q_j} \right) = 0$

Also $[X, C]_{q,p} = \sum_j \left(\frac{\partial X}{\partial q_j} \frac{\partial C}{\partial p_j} - \frac{\partial X}{\partial p_j} \frac{\partial C}{\partial q_j} \right) = 0$

III. $[X, Y + Z]_{q,p} = \sum_j \left(\frac{\partial X}{\partial q_j} \frac{\partial (Y + Z)}{\partial p_j} - \frac{\partial X}{\partial p_j} \frac{\partial (Y + Z)}{\partial q_j} \right)$

$$= \sum_j \left[\frac{\partial X}{\partial q_j} \left(\frac{\partial Y}{\partial p_j} + \frac{\partial Z}{\partial p_j} \right) - \frac{\partial X}{\partial p_j} \left(\frac{\partial Y}{\partial q_j} + \frac{\partial Z}{\partial q_j} \right) \right]$$

$\Rightarrow$ $[X, Y + Z]_{q,p} = \sum_j \left(\frac{\partial X}{\partial q_j} \frac{\partial Y}{\partial p_j} - \frac{\partial X}{\partial p_j} \frac{\partial Y}{\partial q_j} \right) + \sum_j \left(\frac{\partial X}{\partial q_j} \frac{\partial Z}{\partial p_j} - \frac{\partial X}{\partial p_j} \frac{\partial Z}{\partial q_j} \right)$

$$= [X, Y] + [X, Z]$$

IV. $[X, YZ]_{q,p} = \sum_j \left(\frac{\partial X}{\partial q_j} \frac{\partial (YZ)}{\partial p_j} - \frac{\partial X}{\partial p_j} \frac{\partial (YZ)}{\partial q_j} \right)$

$$= \left[\frac{\partial X}{\partial q_j} \left(Y \frac{\partial Z}{\partial p_j} + Z \frac{\partial Y}{\partial p_j} \right) - \frac{\partial X}{\partial p_j} \left(Z \frac{\partial Y}{\partial q_j} + Y \frac{\partial Z}{\partial q_j} \right) \right]$$

$$= 4 \left[\sum_j \left(\frac{\partial X}{\partial q_j} \frac{\partial Z}{\partial p_j} - \frac{\partial X}{\partial p_j} \frac{\partial Z}{\partial q_j} \right) \right] + Z \left[\sum_j \left(\frac{\partial X}{\partial q_j} \frac{\partial Y}{\partial p_j} - \frac{\partial X}{\partial p_j} \frac{\partial Y}{\partial q_j} \right) \right]$$

$$= Y[X, Z] + Z[X, Y]$$

Also

(i) $[q_i, q_j]_{q,p} = 0$

(ii) $[p_i, p_j]_{q,p} = 0$

(iii) $[q_i, p_j]_{q,p} = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

Solution:-

(i) $[q_i, q_j]_{q,p} = \sum_k \left[\frac{\partial q_i}{\partial q_k} \frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial q_j}{\partial q_k} \right]$ ...(1)

Because q_i or q_j is not function of p_k

$$\Rightarrow \quad \frac{\partial q_i}{\partial p_k} = 0, \qquad \frac{\partial q_j}{\partial p_k} = 0$$

(1) $\Rightarrow$ $[q_i, q_j]_{q,p} = 0.$

(ii) $$[p_i, p_j]_{q,p} = \sum_k \left[\frac{\partial p_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\frac{\partial p_j}{\partial q_k} \right]$$

As p_i, p_j is not a function of q_k

$$\therefore \quad \frac{\partial p_i}{\partial q_k} = 0, \qquad \frac{\partial p_j}{\partial q_k} = 0$$

$$\Rightarrow \quad [p_i, p_j]_{q,p} = 0$$

(iii) Now $$[q_i, p_j]_{q,p} = \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k} \right)$$

$$= \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - 0 \right) = \sum_k \frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k}$$

$$= \sum_k \delta_{ik}\,\delta_{jk} = \sum_k \delta_{ij} = \delta_{ij}$$

$$\Rightarrow \quad [q_i, p_j]_{q,p} = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Some other properties:-

If $[\phi, \psi]$ be the Poisson Bracket of ϕ & ψ, then

(1) $$\frac{\partial}{\partial t}[\phi, \psi] = \left[\frac{\partial \varphi}{\partial t}, \psi\right] + \left[\varphi, \frac{\partial \psi}{\partial t}\right]$$

(2) $$\frac{d}{dt}[\phi, \psi] = \left[\frac{d\varphi}{dt}, \psi\right] + \left[\varphi, \frac{d\psi}{dt}\right]$$

Solution:- (1) $$\frac{\partial}{\partial t}[\phi, \psi] = \frac{\partial}{\partial t}\left[\sum_i \left(\frac{\partial \varphi}{\partial q_i}\frac{\partial \psi}{\partial p_i} - \frac{\partial \varphi}{\partial p_i}\frac{\partial \psi}{\partial q_i} \right) \right]$$

$$= \sum_i \frac{\partial}{\partial t}\left[\frac{\partial \varphi}{\partial q_i}\frac{\partial \psi}{\partial p_i} - \frac{\partial \varphi}{\partial p_i}\frac{\partial \psi}{\partial q_i} \right]$$

$$= \sum_i \left[\frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial q_i}\right)\left(\frac{\partial \psi}{\partial p_i}\right) + \frac{\partial \varphi}{\partial q_i}\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial p_i}\right)\right]$$

$$- \left[\frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial p_i}\right)\frac{\partial \psi}{\partial q_i} + \frac{\partial \varphi}{\partial p_i}\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial q_i}\right)\right]$$

$$\sum_i \left[\frac{\partial}{\partial q_i}\frac{\partial \varphi}{\partial t}\left(\frac{\partial \psi}{\partial p_i}\right) - \frac{\partial \psi}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial \varphi}{\partial t}\right)\right] + \frac{\partial \varphi}{\partial q_i}\frac{\partial}{\partial p_i}$$

$$= \sum_i \left[\frac{\partial}{\partial q_i}\left(\frac{\partial \varphi}{\partial t}\right)\frac{\partial \psi}{\partial p_i} + \frac{\partial \varphi}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial \psi}{\partial t}\right) - \frac{\partial}{\partial q_i}\left(\frac{\partial \psi}{\partial t}\right)\left(\frac{\partial \varphi}{\partial p_i}\right) - \frac{\partial \psi}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial \varphi}{\partial t}\right)\right]$$

$$= \sum_i \left[\frac{\partial}{\partial q_i}\left(\frac{\partial \varphi}{\partial t}\right)\frac{\partial \psi}{\partial p_i} - \frac{\partial \psi}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial \varphi}{\partial t}\right)\right]$$

$$+ \sum_i \left[\frac{\partial \varphi}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial \psi}{\partial t}\right) - \frac{\partial \varphi}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{\partial \psi}{\partial t}\right)\right]$$

$$\Rightarrow \quad \frac{\partial}{\partial t}[\phi, \psi] = \left[\frac{\partial \varphi}{\partial t}, \psi\right] + \left[\varphi, \frac{\partial \psi}{\partial t}\right].$$

(2) Similarly, we can prove

$$\frac{d}{dt}[\phi, \psi] = \left[\frac{d\varphi}{dt}, \psi\right] + \left[\varphi, \frac{d\psi}{dt}\right]$$