M.Sc. II sem. (Mathematics)

Paper – Lebesgue Measure & Integration

Topic – Convex Functions & Jenson Inequality

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Convex Functions

Definition. A function ϕ defined an open interval (a, b) is said to be **convex** if for each x, y ε (a, b) and λ, μ such that $\lambda, \mu \geq 0$ and $\lambda + \mu = 1$, we have

$$\phi(\lambda x + \mu y) \leq \lambda\phi(x) + \mu\phi(y)$$

The end points a, b can take the values $-\infty, \infty$ respectively.

If we take $\mu = 1-\lambda$, $\lambda \geq 0$, then $\lambda + \mu = 1$ and so ϕ will be convex if

(5.1.1) $$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$$

If we take $a < s < t < u < b$ and

$$\lambda = \frac{t-s}{u-s}\ ,\ \mu = \frac{u-t}{u-s}\ ,\ u = x,\ s = y\ ,$$

then

$$\lambda + \mu = \frac{t-s+u-t}{u-s} = \frac{u-s}{u-s} = 1$$

and so (5.1.1) reduces to

$$\phi\left(\frac{t-s}{u-s}u + \frac{u-t}{u-s}s\right) \leq \frac{t-s}{u-s}\phi(u) + \frac{u-t}{u-s}\phi(s)$$

or

(5.1.2) $$\phi(t) \leq \frac{t-s}{u-s}\phi(u) + \frac{u-t}{u-s}\phi(u)$$

Thus the segment joining (s, ϕ(s)) and (u, ϕ(n)) is never below the graph of ϕ.
A function ϕ is sometimes said to be convex on (a,b) it for all x, y ε (a, b),

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y)$$

(Clearly this definition is consequence of major definition taking $\lambda = \mu = \frac{1}{2}$).

If for all positive numbers λ, μ satisfying $\lambda + \mu = 1$, we have

$$\phi(\lambda x + \mu y) < \lambda\phi(x) + \mu\phi(y),$$

then ϕ is said to be **Strictly Convex.**

Theorem 5. Let ϕ be convex on (a,b) and $a < s < t < u < b$, then

$$\frac{\phi(t)-\phi(s)}{t-s} \leq \frac{\phi(u)-\phi(s)}{u-s} \leq \frac{\phi(u-\phi(t)}{u-t}$$

If ϕ is strictly convex, equality will not occur.

Proof. Let $a < s < t < u < b$ and suppose ϕ is convex on (a,b). Since

$$\frac{t-s}{u-s} + \frac{u-t}{u-s} = \frac{t-s+u-t}{u-s} = \frac{u-s}{u-s} = 1,$$

therefore, convexity of ϕ yields

$$\phi\left(\frac{t-s}{u-s}u + \frac{u-t}{u-s}s\right) \leq \frac{t-s}{u-s}\phi(u) + \frac{u-t}{u-s}\phi(s)$$

or

(5.1.3) $$\phi(t) \leq \frac{t-s}{u-s}\phi(u) + \frac{u-t}{u-s}\phi(s)$$

or

$$(u-s)\,\phi(t) \leq (t-s)\,\phi(u) + (u-t)\,\phi(s)$$

or

$$(u-s)\,(\phi(t) - \phi(s)) \leq (t-s)\,\phi(u) + u\phi(s) - t\phi(s) - u\phi(s) + s\,\phi(s)$$

or

$$(u-s)(\phi(t) - \phi(s)) \leq (t-s)\,(\phi(u) - \phi(s))$$

or

(5.1.4) $$\frac{\phi(t)-\phi(s)}{t-s} \leq \frac{\phi(u)-\phi(s)}{u-s}$$

This proves the first inequality. The second inequality can be proved similarly.
If ϕ is strictly converse, equality shall not be there in (5.1.3) and so it cannot be in (5.1.4). This completes the pro
of the theorem.

Theorem 6. A differentiable function ϕ is convex on (a,b) if and only if ϕ' is a monotonically increasing function.
ϕ'' exists on (a,b), then ϕ is convex if and only if $\phi'' \geq 0$ on (a, b) and strictly convex if $\psi'' > 0$ on (a,b).

Proof. Suppose first that ϕ is differentiable and convex and let $a < s < t < u < v < b$. Then applying Theorem 5 to $a < s < t < u$, we get

$$\frac{\phi(t)-\phi(s)}{t-s} \leq \frac{\phi(u)-\phi(s)}{u-s} \leq \frac{\phi(u)-\phi(t)}{u-t}$$

and applying Theorem 5 to $a < t < u < v$, we get

$$\frac{\phi(u)-\phi(t)}{u-t} \leq \frac{\phi(v)-\phi(t)}{v-t} \leq \frac{\phi(v)-\phi(v)}{v-u}$$

Hence

$$\frac{\phi(t)-\phi(s)}{t-s} \leq \frac{\phi(v)-\phi(u)}{v-u}$$

If $t \to s$, $\frac{\phi(t)-\phi(s)}{t-s}$ decreases to $\phi'(s)$ and if $u \to v$, $\frac{\phi(v)-\phi(u)}{v-u}$ increases to $\phi'(v)$. Hence $\phi'(v) \geq \phi'(s)$ for all $s < v$ and so ϕ' is monotonically increasing function.

Further, if ϕ'' exists, it can never be negative due to monotonicity of ϕ'.

Conversely, let $\psi'' \geq 0$. Our aim is to show that ψ is convex. Suppose, on the contrary, that ϕ is not convex on (a, b). Therefore, there are points $a < s < t < u < b$ such that

$$\frac{\phi(t)-\phi(s)}{t-s} > \frac{\phi(u)-\phi(t)}{u-t}$$

that is, slope of chord over (s,t) is larger than the slope of the chord over (t,u). But slope of the chord over (s,t) is equal to $\phi'(\alpha)$, for some $\alpha \in (s, t)$ and slope of the chord over (t,u) is $\phi'(\beta)$, $\beta \in (t,u)$. But $\phi'(\alpha) > \phi'(\beta)$ implies ϕ' is not monotone increasing and so ψ'' cannot be greater than zero. We thus arrive at a contradiction. Hence ϕ is convex.

If $\phi'' > 0$, then ϕ is strictly convex, for otherwise there would exist collinear points of the graph of ϕ and we would have $\phi'(\alpha) = \phi'(\beta)$ for appropriate α and β with $\alpha < \beta$. But then $\phi'' = 0$ at some point between α and β which is a contradiction to $\phi'' > 0$. This completes the proof.

Theorem 7. If ϕ is convex on (a,b), then ϕ is absolutely continuous on each closed subinterval of (a,b).

Proof. Let $[c,d] \subset (a,b)$. If $x, y \in [c, d]$, then we have $a < c \leq x \leq y \leq d < b$ and so by Theorem 5, we have

$$\frac{\phi(c)-\phi(a)}{c-a} \leq \frac{\phi(y)-\phi(x)}{y-x} \leq \frac{\phi(b)-\phi(d)}{b-d}$$

Thus

$$|\phi(y) - \phi(x)| \leq M|x-y| \quad , \quad x, y \in [c, d]$$

and so ϕ is absolutely continuous there.

Theorem 8. Every convex function on an open interval is continuous.

Proof. If $a < x_1 < x < x_2 < b$, the convexity of a function ϕ implies

(5.1.5) $$\phi(x) \leq \frac{x_2 - x}{x_2 - x_1}\phi(x_1) + \frac{x - x_1}{x_2 - x_1}\phi(x_2)$$

If we make $x \to x_1$ in (5.1.5), we obtain $\phi(x_1+ 0) \leq \phi(x_1)$; and if we take $x_2 \to x$ we obtain $\phi(x) \leq \phi(x + 0)$. Hence $\phi(x) = \phi(x+0)$ for all values of x in (a,b). Similarly $\phi(x-0) = \phi(x)$ for all values of x. Hence

$$\phi(x-0) = \phi(x+0) = \phi(x)$$

and so ϕ is continuous.

Definition. Let ϕ be a convex function on (a,b) and $x_0 \in (a,b)$. The line

(5.1.6) $$y = m(x-x_0) + \phi(x_0)$$

through $(x_0, \phi(x_0))$ is called a **Supporting Line** at x_0 if it always lie below the graph of ϕ, that is, if

(5.1.7) $$\phi(x) \geq m(x-x_0) + \phi(x_0)$$

The line (5.1.6) is a supporting line if and only if its slope m lies between the left and right hand derivatives at x_0. Thus, in particular, there is at least one supporting line at each point.

Theorem 9 (Jensen Inequality). Let ϕ be a convex function on $(-\infty, \infty)$ and let f be an integrable function on [0,1]. Then

$$\int \phi(f(t))dt \geq \phi[\int f(t)dt]$$

Proof. Put

$$\alpha = \int_0^1 f(t)dt$$

Let $y = m(x-\alpha) + \phi(\alpha)$ be the equation of supporting line at α. Then (by (….) above),

$$\phi(f(t)) \geq m(f(t)-\alpha) + \phi(\alpha)$$

Integrating both sides with respect to t over [0, 1], we have

$$\int_0^1 \phi(f(t))dt \geq m[\int f(t)dt - \int f(t)dt] + \int_0^1 \phi(\alpha)dt$$

$$= 0 + \phi(\alpha) \int_0^1 dt$$

$$= \phi(\alpha) = \phi[\int_0^1 f(t)dt].$$