(L^p - Space) :- Let $[X, \mathcal{S}, \mu]$ be a measure Space and $p \geq 0$ then we define $L^p(\mu)$ to be the class of measurable funⁿ such that

$$[f : \int |f|^p \, d\mu < \infty]$$

L^p - norm :- Let $f \in L^p(\mu)$ then L^p norm of f is define as

$$\|f\|_p = \left(\int |f|^p \, d\mu\right)^{1/p}$$

Theorem Holder ~~Inequat~~ Inequality :-

Statement :- Let $1 < p < \infty$, $1 < q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$

also let $f \in L^p(\mu)$ and $g \in L^q(\mu)$.

then

$$fg \in L^1(\mu) \text{ and}$$

$$\int |fg| \, d\mu \leq \|f\|_p \cdot \|g\|_q$$

OR.

$$\|fg\|_1 \leq \left(\int |f|^p \, d\mu\right)^{1/p} \cdot \left(\int |g|^q \, d\mu\right)^{1/q}$$

proof In this theorem, we use following lemma

If $a \geq 0$, $b \geq 0$, $\frac{1}{p} + \frac{1}{q} = 1$ where $p > 1, q > 1$

then

$$(a)^{1/p} \cdot (b)^{1/q} \leq \frac{a}{p} + \frac{b}{q} \quad \text{--- (1)}$$

If $||f||_p = 0$ & $||g||_q = 0$ then $f.g = 0$ a.e.

then The result is trivial

Let $||f||_p > 0$ & $||g||_q > 0$

put $a = \frac{|f|^p}{(||f||_p)^p}$, $b = \frac{|g|^q}{(||g||_q)^q}$

put this in (1)

$$\left(\frac{|f|^p}{(||f||_p)^p}\right)^{1/p} \cdot \left(\frac{|g|^q}{(||g||_q)^q}\right)^{1/q} \leq \frac{1}{p}\left(\frac{|f|^p}{(||f||_p)^p}\right) + \frac{1}{q}\left(\frac{|g|^q}{(||g||_q)^q}\right)$$

$$\frac{|f|}{||f||_p} \cdot \frac{|g|}{||g||_q} \leq \frac{1}{p}\frac{|f|^p}{(||f||_p)^p} + \frac{1}{q}\frac{|g|^q}{(||g||_q)^q} \quad (2)$$

$\because f \in L^p(\mu)$, $g \in L^q(\mu)$, then the L.H.S of eq^n (2) is integrable then $fg \in L^1(\mu)$

Integrating eq^n (2)

$$\int \frac{|f| \cdot |g| \, d\mu}{||f||_p ||g||_q} \leq \frac{1}{p}\frac{\int |f|^p d\mu}{(||f||_p)^p} + \frac{1}{q}\frac{\int |g|^q d\mu}{(||g||_q)^q}$$

$$\frac{\int |fg| \, d\mu}{||f||_p \cdot ||g||_q} \leq \frac{1}{p}\frac{(||f||_p)^p}{(||f||_p)^p} + \frac{1}{q}\frac{(||g||_q)^q}{(||g||_q)^q}$$

$$\left\{\begin{array}{l} \because \|f\|_p = \left(\int |f|^p d\mu\right. \\ \therefore (\|f\|_p)^p \geq \int |f|^p d\mu \end{array}\right.$$

$$\leq \frac{1}{p} + \frac{1}{q}$$

$$= 1 \quad \{\text{given}\}$$

$$\int |fg| \, d\mu \leq \|f\|_p \|g\|_q$$

OR. $$\|fg\|_1 \leq \left(\int |f|^p d\mu\right)^{1/p} \left(\int |g|^q d\mu\right)^{q}$$

Note If $p = q = 2$ then

$$\|fg\| \leq \left(\int |f|^2 d\mu\right)^{1/2} \left(\int |g|^2 d\mu\right)^{1/2}$$

This is known as "Cauchy Schwarz inequality

Theorem ("Minkowski's inequality")

State:- Let $p \geq 1$, let $f, g \in L^p(\mu)$ then

$$\left(\int |f+g|^p d\mu\right)^{1/p} \leq \left(\int |f|^p d\mu\right)^{1/p} + \left(\int |g|^p d\mu\right)^{1/p}$$

or. $$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Proof:- If $p = 1$ then result is obvious Let $p > 1$ and assume that

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \text{——} \quad (1)$$

Consider
$$\|f+g\|_p = \left(\int |f+g|^p \, d\mu\right)^{1/p}$$

$$\left(\|f+g\|_p\right)^p = \int |f+g|^p \, d\mu$$

$$= \int |f+g|^{p-1} \cdot |f+g| \, d\mu$$

$$\leq \int |f+g|^{p-1} \cdot |f|+|g| \, d\mu$$

$\{\because |A+B| \leq |A|+|B|\}$

$$\leq \int |f+g|^{p-1} |f| d\mu + \int |f+g|^{p-1} |g| d\mu$$

$$\leq \|f\|_p \cdot \|(f+g)^{p-1}\|_q + \|g\|_p \cdot \|(f+g)^{p-1}\|_q$$

$\left\{ \text{By Holder inequality} \right.$

$\left. \int |f| \cdot |g| \, d\mu \leq \|f\|_p \|g\|_q \right\}$

Now,
$$\|(f+g)^{p-1}\|_q = \left(\int |(f+g)^{p-1}|^q \, d\mu\right)^{1/q}$$

$$= \left(\int |(f+g)^{(p-1)q} \, d\mu|\right)^{1/q}$$

$$= \left(\int |f+g|^p \, d\mu\right)^{1/q}$$

$\left\{ \because \frac{1}{p} + \frac{1}{q} = 1 \right.$

$p+q = pq$

$p = pq - q$

$p = (p-1)q$

$$= \left(\left(\int |(f+g)|^{p} d\mu \right)^{1/p} \right)^{1/q}$$

$$\| (f+g)^{p-1} \|_q = \left(\|f+g\|_p \right)^{p/q}$$

$$\left\{ \because \|f\|_p = \left(\int |f|^p d\mu \right)^{1/p} \right\}$$

put this in (2)
we get

$$\left(\|f+g\|_p \right)^p \leq \left(\|f\|_p + \|g\|_p \right) \left(\|f+g\|_p \right)^{p/q}$$

$$\frac{\left(\|f+g\|_p \right)^p}{\left(\|f+g\|_p \right)^{p/q}} \leq \|f\|_p + \|g\|_p$$

$$\left(\|f+g\|_p \right)^{p-p/q} \leq \|f\|_p + \|g\|_p$$

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

$$\left\{ \because \frac{1}{p} + \frac{1}{q} = 1 \right.$$

$$1 + \frac{p}{q} = p$$

$$1 = p - \frac{p}{q}$$

Show that the following inequality are inconsistent for fun^n $f \in L^2[0,\pi]$

$$\left.\begin{aligned} &\int_0^{\pi} [f(x) - \sin x]^2 dx \le \frac{4}{9} \\ &\int_0^{\pi} [f(x) - \cos x]^2 dx \le \frac{1}{9} \end{aligned}\right\} \quad \text{--- (1)}$$

$$\|\sin x - \cos x\|_2 = \|\sin x - f(x) + f(x) - \cos x\|_2$$

$$\le \|\sin x - f(x)\|_2 + \|f(x) - \cos x\|_2$$

{by minkowski's inequality

$$\le \left(\int_0^{\pi} |\sin x - f(x)|^2 dx\right)^{1/2} + \left(\int_0^{\pi} |f(x) - \cos x|^2 dx\right)^{1/2}$$

$$\left\{ \because \|f\|_p = \left(\int |f|^p d\mu\right)^{1/p} \right\}$$

$$\le \left(\frac{4}{9}\right)^{1/2} + \left(\frac{1}{9}\right)^{1/2} \quad \text{by (1)}$$

$$\le \frac{2}{3} + \frac{1}{3} = \cancel{\frac{3}{3}} \; 1$$

ie
$||\sin x - \cos x||_2 \leq 1$ ——— (2)

Again, $||\sin x - \cos x||_2 = \left(\int |\sin x - \cos x|^2 dx\right)^{1/2}$

$= \left(\int_0^\pi |\sin^2 x + \cos^2 x - 2\sin x \cos x| dx\right)^{1/2}$

$\left(\int_0^\pi (1 - \sin 2x) dx\right)^{1/2}$

$= \left[\left(x + \frac{\cos 2x}{2}\right)_0^\pi\right]^{1/2}$

$= \left\{\left[\pi + \frac{\cos 2\pi}{2}\right] - \left[0 + \frac{\cos 0}{2}\right]\right\}^{1/2}$

$\Rightarrow \left(\pi + \frac{1}{2} - \frac{1}{2}\right)^{1/2}$ $\{\cos n\pi = (-1)^n\}$

$||\sin x - \cos x||_2 = \sqrt{\pi}$ ——— (3)

By eq^n (2) & (3) the given inequality are inconsistent.

ue Show that $\int_0^\pi x^{-1/4} \sin x \, dx \leq \pi^{3/4}$

lio Let $f(x) = x^{-1/4}$ and $g(x) = \sin x$

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Then $\int_0^{\pi} f(x)\, g(x)\, dx \le \left(\int_0^{\pi} (f(x))^2 dx\right)^{1/2} \left(\int_0^{\pi} (g(x))^2 dx\right)^{1/2}$

by schortz inequality

$$\le \left(\int_0^{\pi} (x^{-1/4})^2 dx\right)^{1/2} \left(\int_0^{\pi} \sin^2 x\, dx\right)^{1/2}$$

$$\le \left(\int_0^{\pi} (x^{-1/2} dx\right)^{1/2} \cdot \left(\int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right) dx\right)^{1/2}$$

$$\le \left\{\left[\frac{x^{-1/2+1}}{\frac{1}{2}+1}\right]_0^{\pi}\right\}^{1/2} \left\{\left[\frac{1}{2}\left(\frac{x - \sin 2x}{2}\right)\right]_0^{\pi}\right\}^{1/2}$$

$$\le \left[\left(\frac{x^{1/2}}{1/2}\right)_0^{\pi}\right]^{1/2} \cdot \left[\frac{1}{2}\left(\pi - \frac{\sin 2\pi}{2}\right)\right]^{1/2}$$

$$\int_0^{\pi} x^{-1/4} \sin x\, dx \le 2\left[(\pi)^{1/2}\right]^{1/2} \frac{1}{2^{1/2}} (\pi - 0)^{1/2}$$

$$\le 2^{1/2} \cdot (\pi^{1/4}) \frac{1}{2^{1/2}} \pi^{1/2}$$

$$\le \pi^{3/4}$$