Dr. Apurva Muley (Guest Lecturer)

School of Studies in Physics, Vikram University, Ujjain

Lecture for M. Sc. Physics II Semester students

Paper-II: Classical Mechanics

Unit-4 Hamilton's Characteristic Function

7

Hamilton's - Characterstic Function -

OR

Hamilton - ~~Jako~~ Jacobi Eq. for Hamilton's characteristic function -

Conservative system - For conservative system H does not depend on time t we have H as constant (α-says) then the total energy of the system is defined by the Hamilton principle function 'S' becomes -

$$H\left(q_i, \frac{\delta S}{\delta q_i}\right) + \frac{\delta S}{\delta t} = 0 \quad \text{——— (1)}$$

Let us assume the solu. of the form

$$S(q_i, \alpha_i, t) = W(q_i, \alpha_i) - \alpha_i t$$

From which it follows that

$$\frac{\delta S}{\delta q_i} = \frac{\delta W}{\delta q_i} \quad \text{and} \quad \frac{\delta S}{\delta t} = -\alpha_i$$

Putting this value in eq. (1)

$$H\left(q_i, \frac{\delta W}{\delta q_i}\right) = \alpha_i$$

Which is indepandent of time.

Physical Signnificance of W (Hamilton's characteristic function) -

$$W = W(q_i, \alpha_i)$$

So that the time derivative

$$\frac{dW}{dt} = \sum_i \frac{\delta W}{\delta q_i} \dot{q}_i + \sum_j \frac{\delta W}{\delta \alpha_i} \frac{d\alpha_i}{dt}$$

$$= \sum_i \frac{\delta W}{\delta q_i} \dot{q}_i \quad (\because \alpha_i = \text{constant})$$

So that,

$$\frac{dW}{dt} = \sum_i P_i \dot{q}_i$$

$$W = \int \sum_i (P_i, \dot{q}_i) dt$$

$$= \text{action } A$$

Thus W is identified as action A funct?, W is called Hamilton's characteristic funct?.

Hamilton's Eq. of Motion OR Hamilton's Canonical Eq. of Motion –

To obtain Hamilton's eq. of motion for a system with 'n' degrees of freedom, we shall assume that the constraints are holonomic & the forces are derivable from potentials which depend either on position or are velocity-dependent.

In Hamiltonian formulat?, we provide generalised momenta, an independent status, placing it on equal footing with the generalised co-ordinates. Hamiltonian is then to be regarded, in general, as the funct? of the position co-ordinates q_i, momenta p_i & time t.

W.K.T. the Hamiltonian function which is related to the Lagrangian funct? by the eq.

$$H = \sum p_i \dot{q}_i - L(q_i, \dot{q}_i, t) \quad \text{—— (A)} \quad (\because H = T - L)$$

& Hamiltonian can be ~~represen~~ expressed as,

$$H = H(q_i, p_i, t) \qquad (\partial L/\partial \dot{q}_i = p_i)$$

then, the differential form of the above eq. is given by

$$dH = \sum_i \frac{\partial H}{\partial q_i} dq_i + \sum_i \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt \quad \text{—— (1)}$$

but from the definition of H quoted above, we get

$$dH = \sum_i \dot{q}_i \, dp_i + \sum_i p_i \, d\dot{q}_i - dL \quad \text{— (B)}$$

Simillarly, $dL = \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt$

— (2) $[\because L = L(q_i, \dot{q}_i, t)]$

putting the value of dL in eq. (B), we get,

$$dH = \sum_i \dot{q}_i \, dp_i + \sum_i p_i \, d\dot{q}_i - \sum_i \frac{\partial L}{\partial q_i} dq_i - \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial t} dt \quad \text{— (4)}$$

By the definition of generalised momenta, w.k.t.

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \& \quad \dot{p}_i = \frac{\partial L}{\partial q_i}$$

then our eq. (4) becomes,

$$dH = \sum_i \dot{q}_i \, dp_i + \cancel{\sum_i p_i \, d\dot{q}_i} - \sum_i \dot{p}_i \, dq_i - \cancel{\sum_i p_i \, d\dot{q}_i} - \frac{\partial L}{\partial t} dt$$

$$dH = \sum_i \dot{q}_i \, dp_i - \sum_i \dot{p}_i \, dq_i - \frac{\partial L}{\partial t} dt \quad \text{— (5)}$$

On comparing the cofficients of dp_i, dq_i & dt of eq. (1) & (5); we get

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i \quad \text{— (i)} \quad ; \quad \frac{\partial H}{\partial p_i} = \dot{q}_i \quad \text{— (ii)}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad \text{— (iii)}$$

Eq. (i) (ii) & (iii) (are called) known as Hamilton's eq. or Hamilton's canonical eq. of motion.

They constitute a set of $2n$ first-order differential eq. of motion replacing the Langrange eq. of II order.