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Lecture for M. Sc. Physics II Semester students

Paper-II: Classical Mechanics

Unit-4 Canonical Transformation

CANONICAL TRANSFORMAT^N :-

Sometimes it is convinient to change the old variable (q_K, p_K) to new variable (Q_K, P_K)

There are number of problem in mechanics which desire to change one set of position and momentum coordinates into another set of position and momentum coordinates. which may be rather suitable.

Let us suppose (p_K, q_K) are old set of position and momentum coordinates and (P_K, Q_K) are new set of momentum and posi(n) coordinate. which are defined as-

$$\left.\begin{aligned} P_K &= P_K(p_K, q_K, t) \\ Q_K &= Q_K(p_K, q_K, t) \end{aligned}\right] \text{——} (1)$$

By Hamiltonian,

$$\left.\begin{aligned} \dot{P}_K &= -\frac{\partial \bar{H}}{\partial Q_K} \\ \dot{Q}_K &= \frac{\partial \bar{H}}{\partial P_K} \end{aligned}\right] \text{——} (2)$$

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So, the transformation for which Eq. (2) are valid called canonical transformation. It is called contact transformation.

Let H and L are Lagrangian and Hamiltonian of the old system. and $\bar{H}$ and $\bar{L}$ are Hamiltonian and Lagrangian of New system. of coordinates of position and momentum.

Then,

$$\left.\begin{aligned} H &= \sum_{k=1}^{n} p_k \dot{q}_k - L \\ \bar{H} &= \sum_{k=1}^{n} P_k \dot{Q}_k - \bar{L} \end{aligned}\right\} \text{— (3)} \qquad \left.\begin{aligned} L &= \sum_{k=1}^{n} p_k \dot{q}_k - H \\ \bar{L} &= \sum_{k=1}^{n} P_k \dot{Q}_k - \bar{H} \end{aligned}\right\} \text{— (4)}$$

and Lagrangian for time interval t_1 and t_2,

$$\left.\begin{aligned} \delta \int_{t_1}^{t_2} L\,dt &= 0 \\ \delta \int_{t_1}^{t_2} \bar{L}\,dt &= 0 \end{aligned}\right] \text{— (5)}$$

using eq. (4) into (5), we get.

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$$\left.\begin{aligned}\delta\int_{t_1}^{t_2}\left[\sum_{k=1}^{n} p_K \dot{q}_K - H\right] dt = 0 \\ \delta\int_{t_1}^{t_2}\left[\sum_{k=1}^{n} P_K \dot{Q}_K - \bar{H}\right] dt = 0\end{aligned}\right\} \quad \text{--- (6)}$$

Differentiating eqⁿ (6) by a total time derivatives of an arbitrary funcⁿ say F.

$$\left[\sum_{k=1}^{n} p_K \dot{q}_K - H\right] - \left[\sum_{k=1}^{n} P_K \dot{Q}_K - \bar{H}\right] = \frac{dF}{dt} \quad \text{--- (6a)}$$

Now combining to eq. (6).

$$= \delta\int_{t_1}^{t_2}\left[\left(\sum_{k=1}^{n} p_K \dot{q}_K - H\right) - \left(\sum_{k=1}^{n} P_K \dot{Q}_K - \bar{H}\right)\right] dt = 0$$

$$= \delta\int_{t_1}^{t_2} \frac{dF}{dt}\, dt = 0$$

$$= \delta\int_{t_1}^{t_2} dF = 0$$

$$= \delta\, |F|_{t_1}^{t_2} = 0$$

$$= \delta\,[F(t_2) - F(t_1)] = 0$$

The funcⁿ F is called Generating funcⁿ of transformation, where, F is an general funcⁿ of $(4n+1)$ variables — (q_K, p_K, Q_K, P_K, t).

where $K = 1 - - n$.

Then function f can be reduced into a function of $(2n+1)$ independent variables of which one is t and other $2n$ are from q_K, p_K, Q_K, P_K.

There are following form of Generating funcⁿ F.

$$\left.\begin{aligned} F_1 &= F_1(q_K, Q_K, t) \\ F_2 &= F_2(q_K, P_K, t) \\ F_3 &= F_3(p_K, Q_K, t) \end{aligned}\right\} \circ$$

$$F_4 = F_4(p_K, P_K, t).$$

I:- $F = F_1(q_K, Q_K, t)$. —— (1)

When generating function F is the function of old and new position coordinates.

Then,

$$\frac{dF}{dt} = \frac{dF_1}{dt} = \sum_{K=1}^{n} \frac{\partial F_1}{\partial q_K} \cdot \frac{dq_K}{dt} + \sum_{K=1}^{n} \frac{\partial F_1}{\partial Q_K} \cdot \frac{dQ_K}{dt} + \frac{\partial F_1}{\partial t}.$$

$$= \sum_{K=1}^{n} \frac{\partial F_1}{\partial q_K} \cdot \dot{q}_K + \sum_{K=1}^{n} \frac{\partial F_1}{\partial Q_K} \dot{Q}_K + \frac{\partial F_1}{\partial t} \quad \text{—— (2)}$$

Now By definition of generating funcn :-

$$\frac{dF}{dt} = \left[\left(\sum_{K=1}^{n} p_K \dot{q}_K - H\right) - \left(\sum_{K=1}^{n} P_K \dot{Q}_K - \bar{H}\right)\right] \quad \text{--- (3)}$$

From (2) and (3).

$$\left[\left(\sum_{K=1}^{n} p_K \dot{q}_K - H\right) - \left(\sum_{K=1}^{n} P_K \dot{Q}_K - \bar{H}\right)\right] =$$

$$\sum_{K=1}^{n} \frac{\partial F_1}{\partial q_K} \dot{q}_K + \sum_{K=1}^{n} \frac{\partial F_1}{\partial Q_K} \dot{Q}_K + \frac{\partial F_1}{\partial t}$$

or.

$$\sum_{K=1}^{n} \left(p_K - \frac{\partial F_1}{\partial q_K}\right) \dot{q}_K - \sum_{K=1}^{n} \left(P_K + \frac{\partial F_1}{\partial Q_K}\right) \dot{Q}_K + \bar{H} - H - \frac{\partial F_1}{\partial t} = 0$$

$$\sum_{K=1}^{n} \left(p_K - \frac{\partial F_1}{\partial q_K}\right) \frac{dq_K}{dt} - \sum_{K=1}^{n} \left(P_K + \frac{\partial F_1}{\partial Q_K}\right) \frac{dQ_K}{dt} + \left(\bar{H} - H - \frac{\partial F_1}{\partial t}\right) = 0$$

$$\sum_{K=1}^{n} \left(p_K - \frac{\partial F_1}{\partial q_K}\right) dq_K - \sum_{K=1}^{n} \left(P_K + \frac{\partial F_1}{\partial Q_K}\right) dQ_K + \left(\bar{H} - H - \frac{\partial F_1}{\partial t}\right) dt = 0$$

Treating q_K, Q_K, t as independent variables

and equating the coefficient of each term of either side is zero.

$$p_K - \frac{\partial F_1}{\partial q_K} = 0 \Rightarrow \boxed{p_K = \frac{\partial F_1}{\partial q_K}}$$

$$P_K + \frac{\partial F_1}{\partial Q_K} = 0 \Rightarrow \boxed{P_K = -\frac{\partial F_1}{\partial Q_K}} \quad \text{—— (4)}$$

$$\bar{H} - H - \frac{\partial F_1}{\partial t} \Rightarrow \boxed{\bar{H} - H = \frac{\partial F_1}{\partial t}}$$

Above Eq. are canonical transformation.

II $F_2 = F_2(q_K, P_K, t)$ —— (2)

When generating function is a function of old position and new momentum coordinate

On comparing with F_1 we can see that F_2 can be obtained from F_1 by replacing Q_K by P_K. Therefore by keeping in Legendre transformation we can obtained F_2 from F_1

$$F_2(q_k, P_k, t) = F_1(q_k, Q_k, t) + \sum_{k=1}^{n} P_k Q_k$$

$$F_1 = F_2(q_k, P_k, t) - \sum_{k=1}^{n} P_k Q_k$$

$$\frac{dF_1}{dt} = \frac{dF_2}{dt}(q_k, P_k, t) - \frac{d}{dt}\left(\sum_{k=1}^{n} P_k Q_k\right)$$

$$= \sum_{k=1}^{n} \frac{\partial F_2}{\partial q_k} \cdot \frac{dq_k}{dt} + \sum_{k=1}^{n} \frac{\partial F_2}{\partial P_k} \frac{dP_k}{dt} + \frac{\partial F_2}{\partial t} - \sum_{k=1}^{n} P_k \dot{Q}_k - \sum_{k=1}^{n} \dot{P}_k Q_k$$

$$= \sum_{k=1}^{n} \frac{\partial F_2}{\partial q_k} \dot{q}_k + \sum_{k=1}^{n} \frac{\partial F_2}{\partial P_k} \dot{P}_k + \frac{\partial F_2}{\partial t} - \sum_{k=1}^{n} P_k \dot{Q}_k - \sum_{k=1}^{n} \dot{P}_k Q_k \quad \text{--- (5)}$$

But from the definitn of generating funcn

$$\frac{dF}{dt} = \frac{dF_1}{dt} = \left[\sum_{k=1}^{n} p_k \dot{q}_k - H\right] - \left[\sum_{k=1}^{n} P_k \dot{Q}_k - \bar{H}\right] \quad \text{--- (6)}$$

comparing from eq (5) and (6)

$$\sum_{k=1}^{n}(p_k \dot{q}_k - H) - \sum_{k=1}^{n}\left[\cancel{P_k \dot{Q}_k} - \bar{H}\right] = \sum_{k=1}^{n} \frac{\partial F_2}{\partial q_k} \dot{q}_k + \sum_{k=1}^{n} \frac{\partial F_2}{\partial P_k} \dot{P}_k + \frac{\partial F_2}{\partial t} - \cancel{\sum_{k=1}^{n} P_k \dot{Q}_k} - \sum_{k=1}^{n} \dot{P}_k Q_k$$

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$$\sum_{K=1}^{n}\left(p_K - \frac{\partial f_2}{\partial q_K}\right)\dot{q}_K - \sum_{K=1}^{n}\left(\frac{\partial f_2}{\partial P_K} - Q_K\right)\dot{P}_K + \bar{H} - H - \frac{\partial f_2}{\partial t} = 0$$

$$\sum_{K=1}^{n}\left(p_K - \frac{\partial f_2}{\partial q_K}\right)\frac{dq_K}{dt} - \sum_{K=1}^{n}\left(\frac{\partial f_2}{\partial P_K} - Q_K\right)\frac{dP_K}{dt} + \left[\bar{H} - H - \frac{\partial f_2}{\partial t}\right] = 0$$

$$\sum_{K=1}^{n}\left(p_K - \frac{\partial f_2}{\partial q_K}\right)dq_K - \sum_{K=1}^{n}\left(\frac{\partial f_2}{\partial P_K} - Q_K\right)dP_K + \left(\bar{H} - H - \frac{\partial f_2}{\partial t}\right)dt$$

$\therefore$ since q_K, P_K and t are independent varia coordinate so their coeff. must be vanish separately.

$$p_K - \frac{\partial f_2}{\partial q_K} = 0 \Rightarrow \boxed{p_K = \frac{\partial f_2}{\partial q_K}} \quad \text{—— (7a)}$$

$$\frac{\partial f_2}{\partial P_K} - Q_K = 0 \Rightarrow \boxed{Q_K = \frac{\partial f_2}{\partial P_K}} \quad \text{—— (7b)}$$

$$\bar{H} - H - \frac{\partial f_2}{\partial t} \Rightarrow \boxed{\bar{H} - H = \frac{\partial f_2}{\partial t}}$$

Eq. (7a) and (7b) are canonical eq^n for generating func^n $F_2(q_K, P_K, t)$

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III $F_3 = F_3(p_K, Q_K, t)$ —— (3)

when generating funcn is the funcn of old momentum and new position coordinate i.e.

The funcn F_3 can be obtained from generating funcn F_1 by replacing q_K by p_K. So it is obtained by Legendre's transformation

$$F_3 = F_3(p_K, Q_K, t) = F_1(q_K, Q_K, t) - \sum_{K=1}^{n} p_K q_K$$

$$F_1 = F_3(p_K, Q_K, t) + \sum_{K=1}^{n} p_K q_K$$

$$\frac{dF_1}{dt} = \sum_{K=1}^{n} \frac{\partial F_3}{\partial p_K} \dot{p}_K + \sum_{K=1}^{n} \frac{\partial F_3}{\partial Q_K} \dot{Q}_K + \frac{\partial F_3}{\partial t} + \sum_{K=1}^{n} p_K \dot{q}_K + \sum_{K=1}^{n} \dot{p}_K q_K \quad (8)$$

Now from definition of GF -

$$\frac{dF}{dt} = \frac{dF_1}{dt} = \left[\sum_{K=1}^{n} p_K \dot{q}_K - H\right] - \left(\sum_{K=1}^{n} P_K \dot{Q}_K - \bar{H}\right) \quad (9)$$

using Eq. 8 and 9

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$$\sum_{k=1}^{n} (\cancel{p_k \dot{q}_k} - H) - \sum_{k=1}^{n} (P_K \dot{Q}_K - \bar{H}) =$$

$$\sum_{k=1}^{n} \frac{\partial F_3}{\partial p_k} \dot{p}_k + \sum_{k=1}^{n} \frac{\partial F_3}{\partial Q_K} \dot{Q}_K + \frac{\partial F_3}{\partial t} +$$

$$\sum_{k=1}^{n} \cancel{p_k \dot{q}_k} + \sum_{k=1}^{n} \dot{p}_k q_k$$

$$\Rightarrow \sum_{k=1}^{n} \left(\frac{\partial F_3}{\partial Q_K} + P_K\right) \dot{Q}_K + \sum_{k=1}^{n} \left(\frac{\partial F_3}{\partial p_k} + q_k\right) \dot{p}_k + \frac{\partial F_3}{\partial t}$$

$$+ \bar{H} + H = 0$$

$$\Rightarrow \sum_{k=1}^{n} \left(\frac{\partial F_3}{\partial Q_K} + P_K\right) dQ_K + \sum_{k=1}^{n} \left(\frac{\partial F_3}{\partial p_k} + q_k\right) dp_k +$$

$$\left(\frac{\partial F_3}{\partial t} - \bar{H} + H\right) dt = 0$$

since Q_K, p_K, t are independent coefficient.

$$P_K + \frac{\partial F_3}{\partial Q_K} = 0 \qquad \Rightarrow P_K = -\frac{\partial F_3}{\partial Q_K} \qquad \text{— (10a)}$$

$$\frac{\partial F_3}{\partial p_k} + q_k = 0 \qquad \Rightarrow q_k = \frac{-\partial F_3}{\partial p_k} \qquad \text{— (10b)}$$

$$\frac{\partial F_3}{\partial t} + H - \bar{H} = 0 \qquad \Rightarrow \frac{\partial F_3}{\partial t} = \bar{H} - H$$

Eq^n (10a) and 10b are canonical eq. for generating func^n (p_K, Q_K, t).

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4. $F_4 = F_4(p_K, P_K, t)$ —— (I)

when a GF is the func^n of old and new momentum coordinate.

The function F_4 can be obtain from func^n F_1 by replacing q_K by p_K and Q_K by P_K.

using of legendres Transformat^n.

$$F_4 = F_4(p_K, P_K, t) = F_1(q_K, Q_K, t) + \sum_{K=1}^{n} P_K Q_K - \sum_{K=1}^{n} q_K p_K$$

Both

$$F_1 = F_4(p_K, P_K, t) + \sum_{K=1}^{n} q_K p_K - \sum_{K=1}^{n} P_K Q_K$$

$$\frac{dF_1}{dt} = \sum_{K=1}^{n} \frac{\partial F_4}{\partial p_K} \dot{p}_K + \sum_{K=1}^{n} \frac{\partial F_4}{\partial P_K} \dot{P}_K + \frac{\partial F_4}{\partial t} + \sum_{K=1}^{n} q_K \dot{p}_K + \sum_{K=1}^{n} \dot{q}_K p_K - \sum_{K=1}^{n} P_K \dot{Q}_K - \sum_{K=1}^{n} \dot{P}_K Q_K \quad \text{—— (II)}$$

By definitn of G.F. :-

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$$\frac{dF}{dt} = \frac{dF_1}{dt} = \left(\sum_{k=1}^{n} p_k \dot{q}_k - H\right) - \left(\sum_{k=1}^{n} P_k \dot{Q}_k - \bar{H}\right) \quad (12)$$

using eq. (11) and (12),

$$\Rightarrow \sum_{k=1}^{n} \cancel{p_k \dot{q}_k} - H - \sum_{k=1}^{n} \cancel{P_k \dot{Q}_k} - \bar{H} =$$

$$\sum_{k=1}^{n} \frac{\partial F_4}{\partial p_k} \dot{p}_k + \sum_{k=1}^{n} \frac{\partial F_4}{\partial P_k} \dot{P}_k + \sum_{k=1}^{n} \frac{\partial F_4}{\partial t} +$$

$$\sum_{k=1}^{n} q_k \dot{p}_k + \sum_{k=1}^{n} \cancel{\dot{q}_k p_k} - \sum_{k=1}^{n} \cancel{P_k \dot{Q}_k} - \sum_{k=1}^{n} \dot{P}_k Q_k$$

$$\Rightarrow \sum_{k=1}^{n} \left(\frac{\partial F_4}{\partial p_k} + q_k\right) \dot{p}_k + \sum_{k=1}^{n} \left(\frac{\partial F_4}{\partial P_k} - Q_k\right) \dot{P}_k + \left(\frac{\partial F_4}{\partial t} + H - \bar{H}\right) = 0$$

$$\sum_{k=1}^{n} \left(\frac{\partial F_4}{\partial p_k} + q_k\right) dp_k + \sum_{k=1}^{n} \left(\frac{\partial F_4}{\partial P_k} - Q_k\right) dP_k$$

$$+ \left(\frac{\partial F_4}{\partial t} - \bar{H} + H\right) dt = 0$$

since p_k, P_k and t are indep. coordinate so their coeff. must be vanish.

$$\frac{\partial F_4}{\partial p_K} + q_K = 0 \Rightarrow \frac{\partial F_4}{\partial p_K} = -q_K \quad \text{——— 13a}$$

$$\frac{\partial F_4}{\partial P_K} - Q_K = 0 \Rightarrow \frac{\partial F_4}{\partial P_K} = Q_K \quad \text{——— 13b}$$

$$\frac{\partial F_4}{\partial t} - \bar{H} + H = 0 \Rightarrow \frac{\partial F_4}{\partial t} = \bar{H} - H$$

Eq^n 13a and 13b are canonical transformation eq^n for generating func^n (p_K, P_K, t).

Eqn of Motion in poisson Bracket form :-

Let f be a funcn depending upon positn, momenta and time.

$$f = f(q, p, t)$$

$$\frac{df}{dt} = \sum_K \frac{\partial f}{\partial q_K} \dot{q}_K + \frac{\partial f}{\partial p_K} \dot{p}_K + \frac{\partial f}{\partial t}.$$

$$\dot{q}_K = \frac{\partial H}{\partial p_K} \text{ and } \dot{p}_K = -\frac{\partial H}{\partial q_K}$$

$$\frac{df}{dt} = \sum_K \frac{\partial f}{\partial q_K} \cdot \frac{\partial H}{\partial p_K} - \sum_K \frac{\partial f}{\partial p_K} \cdot \frac{\partial H}{\partial q_K} + \frac{\partial f}{\partial t}.$$

$$= \sum \left(\frac{\partial f}{\partial q_K} \cdot \frac{\partial H}{\partial p_K} - \frac{\partial f}{\partial p_K} \cdot \frac{\partial H}{\partial q_K} \right) + \frac{\partial f}{\partial t}.$$

$$= [f, H]_{q,p} + \frac{\partial f}{\partial t}.$$

poisson Bracket form. :-

If the f is an constant of mom :-

so, $\frac{df}{dt} = 0$

$$[f, H] + \frac{\partial f}{\partial t} = 0$$

If f does not depend upon time explicity

$\frac{\partial f}{\partial t} = 0$ therefore, $[f, H] = 0$