2. The algebra of matrices

In this section, we shall define a matrix and discuss some of the simple operations by which two or more matrices can be combined.

Definition and notation

A rectangular array of numbers (real or complex) is called a *matrix*. The array is usually enclosed within square or curved brackets. Thus, the rectangular arrays

$$\begin{bmatrix} 2 & -3 \\ 5 & 7 \end{bmatrix}, \begin{bmatrix} 3 & -5 & 9 \\ 8 & 6 & -2 \end{bmatrix}, \begin{bmatrix} a & b & c \\ j & k & l \\ r & s & t \\ x & y & z \end{bmatrix}, \begin{bmatrix} 1+2i & 3-4i \\ 5+id & x+iy \\ s+4it & u+7i \end{bmatrix} \tag{1}$$

are examples of a matrix. The individual members of the array are called the *elements* of the matrix. Although we have defined a matrix here with reference to numbers, we can easily extend the definition to a matrix whose elements are functions. An example of such a matrix is

$$\begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_4(x) & f_5(x) & f_6(x) \end{bmatrix}, \tag{2}$$

where $f_i(x)$ are functions of x.

It is convenient to think of every element of a matrix as belonging to a certain *row* and a certain *column* of the matrix. Thus, referring to the third matrix in (1), we would say that $(a\ b\ c)$ is the first row of the matrix, $(j\ k\ l)$ the second row, $(r\ s\ t)$ the third row and $(x\ y\ z)$ the fourth row of the matrix. Similarly, the matrix has three columns which are

$$\begin{bmatrix} a \\ j \\ r \\ x \end{bmatrix}, \begin{bmatrix} b \\ k \\ s \\ y \end{bmatrix} \text{ and } \begin{bmatrix} c \\ l \\ t \\ z \end{bmatrix}. \tag{3}$$

It is then obvious that every element of a matrix can be uniquely characterized by a row index and a column index. The element s, for example, belongs to the third row and the second column, the element z belongs to the fourth row and the third column, etc.

If a matrix has m rows and n columns, we shall say that the matrix is of order $m \times n$ (called 'm by n'). A general matrix of order $m \times n$ can be conveniently written as

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}, \tag{4}$$

where the elements may be real or complex numbers or functions. This may be condensed to the shorthand notation

$$\mathbf{A}\equiv[a_{ij}]_{m\times n}, \tag{5a}$$

which means that **A** is a matrix of order $m\times n$ whose ij-th element[1] is a_{ij}. It should be clear that $1\leqslant i\leqslant m$ and $1\leqslant j\leqslant n$. If the order of the matrix **A** need not be written explicitly, we may also express this as

$$(\mathbf{A})_{ij}=a_{ij}, \tag{5b}$$

which merely states that the ij-th element of **A** is a_{ij}.

The zero matrix

A matrix **A** of arbitrary order is said to be a *zero matrix* if, and only if, every element of **A** equals zero. We shall denote a zero matrix by **0** (not to be confused with the null element of a vector spaes, which is also denoted by the same symbol). Thus, if **A** is of order $m\times n$, then

$$\mathbf{A}=\mathbf{0}\Leftrightarrow(\mathbf{A})_{ij}=0 \text{ for } 1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n. \tag{6}$$

If it is necessary to specify the order of a zero matrix, we may write it as $\mathbf{0}_{m\times n}$. It is evident that for any arbitrary matrix **A**,

$$\mathbf{A}+(-\mathbf{A})=\mathbf{A}-\mathbf{A}=\mathbf{0}. \tag{7}$$

Combination of matrices

Before we proceed to discuss the combination of matrices and the various identities and relations among matrices, the first logical step would be to define the *equality of two matrices.*

In order that two matrices **A** and **B** be equal to each other, it is necessary, though not sufficient, that they be of the same order. Granted that **A** and **B** are of the same order, say $m\times n$, we say that **A** is equal to **B** *if, and only if,* every element of **A** is equal to the corresponding element of **B**. Thus, if $\mathbf{A}\equiv[a_{ij}]_{m\times n}$ and $\mathbf{B}\equiv[b_{ij}]_{m\times n}$, then

$$\mathbf{A}=\mathbf{B}\Leftrightarrow a_{ij}=b_{ij}, \text{ for } 1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n.$$

[1]An element in the i-th row and the j-th column of a matrix will be ca[lled] the ij-th element of the matrix.

This set of mn conditions extends the concept of equality of numbers to that of equality of matrices.

Matrix addition

We may now define the addition and subtraction of two matrices for which, again, both the matrices must be of the same order. If **A** is the matrix defined earlier and $\mathbf{C}\equiv[c_{ij}]_{m\times n}$, then the sum of **A** and **C** is defined as a matrix of the same order and whose ij-th element equals the sum of the ij-th elements of **A** and **C**. Thus,

$$\mathbf{A}+\mathbf{C}\equiv[a_{ij}+c_{ij}]_{m\times n}, \quad \text{or} \quad (\mathbf{A}+\mathbf{C})_{ij}=a_{ij}+c_{ij}. \tag{9}$$

Similarly, we define the difference of the two matrices **A** and **C** as a matrix of the same order and whose elements are given by

$$\mathbf{A}-\mathbf{C}\equiv[a_{ij}-c_{ij}]_{m\times n}=-(\mathbf{C}-\mathbf{A}), \tag{10a}$$

or

$$(\mathbf{A}-\mathbf{C})_{ij}=-(\mathbf{C}-\mathbf{A})_{ij}=a_{ij}-c_{ij}. \tag{10b}$$

Because of the fact that matrix addition is a simple extension of the concept of addition of numbers, *the law of matrix addition is commutative.* This means that just as any two scalars satisfy the property $a+c=c+a$, for any two matrices whose sum can be defined (that is, which are of the same order), we have

$$\mathbf{A}+\mathbf{C}=\mathbf{C}+\mathbf{A}. \tag{11}$$

EXAMPLE 1: Find the sum of the two matrices

$$\mathbf{A}=\begin{bmatrix} 2 & 5 & 0 & 7 \\ -1 & 6 & 2 & 4 \\ 3 & -4 & 8 & -2 \end{bmatrix}, \quad \mathbf{B}=\begin{bmatrix} -1 & 0 & 3 & -3 \\ 2 & 2 & -2 & 5 \\ -3 & 5 & 6 & 4 \end{bmatrix}.$$

Solution: Both the matrices are of order 3×4, so that their sum is defined and is given by

$$\mathbf{A}+\mathbf{C}=\begin{bmatrix} 2-1 & 5+0 & 0+3 & 7-3 \\ -1+2 & 6+2 & 2-2 & 4+5 \\ 3-3 & -4+5 & 8+6 & -2+4 \end{bmatrix}=\begin{bmatrix} 1 & 5 & 3 & 4 \\ 1 & 8 & 0 & 9 \\ 0 & 1 & 14 & 2 \end{bmatrix}.$$

The concept can obviously be extended to more than two matrices. Thus, for any matrices **A**, **B**, **C**, **D**,..., all of the same order, we have

$$(\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D}+\ldots)_{ij}=(\mathbf{A})_{ij}+(\mathbf{B})_{ij}+(\mathbf{C})_{ij}+(\mathbf{D})_{ij}+\cdots \tag{12}$$

Moreover, it is easy to see that in the triple sum $\mathbf{A}+\mathbf{B}+\mathbf{C}$, we may ther first add **A** to **B** and then add the resulting matrix to **C**, or t add **B** to **C** and then add the resulting matrix to **A**. In mathe- tical notation, we have

$$(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C}). \tag{13}$$

We say that *the law of matrix addition is associative.*

This may seem to be a little trivial, but this is an important property satisfied only by certain laws of higher algebra. To give a simple example, it is easy to see that *the law of matrix subtraction is not associative*; thus,

$$(\mathbf{A}-\mathbf{B})-\mathbf{C}\neq\mathbf{A}-(\mathbf{B}-\mathbf{C}). \tag{14}$$