Unitary matrix

A matrix **U** satisfying the relations

$$\mathbf{U}\mathbf{U}^{\dagger}=\mathbf{I}_n, \quad (39a)$$

$$\mathbf{U}^{\dagger}\mathbf{U}=\mathbf{I}_m, \quad (39b)$$

is called a *unitary matrix*. If **U** is a finite matrix satisfying both Eqs. (39), then **U** must be a square matrix, and[3] $\mathbf{U}\mathbf{U}^{\dagger}=\mathbf{I} \Leftrightarrow \mathbf{U}^{\dagger}\mathbf{U}=\mathbf{I}$. The elements of a unitary matrix may be complex. In fact, it is evident from Eqs. (39) that a real unitary matrix is orthogonal.

Let det $\mathbf{U}=d$. Taking the determinants of both sides of Eq. (39a) and noting that det $\mathbf{U}^{\dagger}=d^*$, we have

$$dd^*=1 \Rightarrow |d|=1. \quad (40)$$

This shows that the determinant of a unitary matrix can be a complex number of unit magnitude, that is, a number of the form $e^{i\theta}$, where θ is a real number. It also shows that a unitary matrix is nonsingular and possesses an inverse. Multiplying Eq. (39a) from the left by $\mathbf{U}^{-1}$, we have

$$\mathbf{U}^{\dagger}=\mathbf{U}^{-1}. \quad (41)$$

Equating the *ij*-th element of both sides of Eq. (39a) and of Eq. (39b), we have

$$\sum_{k=1}^{n} u_{ik}u_{jk}^*=\delta_{ij}, \quad \sum_{k=1}^{n} u_{ki}u_{kj}^*=\delta_{ij}, \quad 1\leqslant i, j\leqslant n, \quad (42)$$

where u_{ik} is the *ik*-th element of **U** and n is its order. These are the conditions satisfied by the elements of a unitary matrix. As in the case of an orthogonal matrix, the sets of conditions of Eqs. (42) are not independent of each other for a finite matrix.