Field

A *field* $F=\{a, b, c, d, \ldots\}$ is a set of elements, endowed with two binary laws of composition for its elements, one denoted by $+$ called *addition*, and the other denoted by $\circ$ called *multiplication*, such that the following two conditions are satisfied:

(a) F is an abelian group under addition with the identity element denoted by 0 and called *zero*; and

(b) the set of the nonzero elements of F is an abelian group under multiplication with the identity element denoted by 1 and called *unity*.

0 is called the *additive identity* while 1 the *multiplicative identity* of the field.

Examples of a field are:

1. The set R of all real numbers with the additive identity 0 and the multiplicative identity 1.
2. The set C of all complex numbers with the additive identity $0+0i$ and the multiplicative identity $1+0i$.
3. The set $\{0, 1, 2, \ldots, p-1\}$ of p integers, where p is a prime number greater than 1, with the two binary operations of addition modulo p and multiplication modulo p; a finite field is called a *Galois field.*

The elements of a field are called *scalars.*

Vector space

A set $L=\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \ldots\}$ is said to be a *vector space over a field F* if the following two conditions are satisfied:

(a) An operation of addition denoted by $+$ is defined in L such that L is an abelian group under addition. The identity element of this group will be denoted by $\mathbf{0}$.

(b) A scalar of the field F and an element of the set L can be combined by an operation called *scalar multiplication* to give an element of L such that for every $\mathbf{u}, \mathbf{v} \in L$ and $a, b \in F$, we have

$$a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}\in L; \quad (a+b)\,\mathbf{u}=a\mathbf{u}+b\mathbf{u}\in L,$$
$$a(b\mathbf{u})=(a \circ b)\,\mathbf{u}, \quad 1\mathbf{u}=\mathbf{u},\ 0\mathbf{u}=\mathbf{0}. \tag{2}$$

The elements of a vector space are called *vectors.* Also note that 0 is an element (the additive identity) of F, whereas $\mathbf{0}$ is the *null vector* of L. The phrases *linear vector space* and *linear space* are also used for a vector space. Henceforth, we shall drop the multiplication symbol for scalars and write, for example, $a \circ b$ simply as ab.

The familiar three-dimensional space of position vectors is an example of a vector space over the field of real numbers. First, it is evident that the set of all position vectors is an abelian group. For, if $\mathbf{u}$ and $\mathbf{v}$ are any two vectors of this space, $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ is also a vector of this space. The identity element is the null vector $\mathbf{0}$. The 'inverse' of a vector $\mathbf{u}$ is the vector $-\mathbf{u}$, because $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$. Moreover, the law of vector addition is associative. Second, the position vectors satisfy the properties listed in Eqs. (2) for all a, b belonging to the field R of real numbers.