Lagrange's Eqn from D' Alembert Principle

$$r_i = r_i(q_1, q_2 \cdots q_n, t) \quad \text{——} (1)$$

So that where t is the time & q_n are the generalized coordinates

Differentiating eqn (1) with respect to t, we obtain the velocity of the ith particle i.e.

$$\frac{dr_i}{dt} = \frac{\partial r_i}{\partial q_1}\frac{dq}{dt} + \frac{\partial r_i}{\partial q_2}\frac{dq_2}{dt} + \cdots \frac{\partial r_i}{\partial t}\frac{dq_n}{dt} + \frac{\partial r_i}{\partial t}$$

or

$$v_i = \dot{r}_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t} \quad \text{——} (2)$$

where $\dot{q}_j$ are the generalized velocities

The virtual displacement is given by

$$\delta r_i = \sum_{i,j} \frac{\partial r_i}{\partial q_j}\delta q_1 + \frac{\partial r_i}{\partial q_2}\delta q_2 + \cdots \frac{\partial r_i}{\partial q_j}\delta q_j + \cdots \frac{\partial r_i}{\partial q_n}\delta q_n \quad \text{——} (3)$$

$$\delta r_i = \sum_{i,j}^{n} \frac{\partial r_i}{\partial q_j}\delta q_j \quad \text{——} (4)$$

{ by the defination the virtual displacements do not depend on time.

According to D' Alembert's Principle

$$\sum_{i=1}^{n} (F_i^{(a)} - \dot{P}_i)\cdot \delta r_i = 0 \quad \text{——} (A)$$

$$\sum_{i=1}^{n} F_i^{(a)}\cdot \delta r_i - \sum_{i=1}^{n} \dot{P}_i \cdot \delta r_i = 0. \quad \text{——} (5)$$

putting the value of δr_i from eqn (4), we get

Here $\sum_{i=1}^{n} F_i^{(a)}\cdot \delta r_i = \sum_{i=1}^{n} F_i^{(a)} \cdot \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j}\delta q_j$

$$= \sum_{i=j=1}^{n} \left[F_i^{(a)} \cdot \frac{\partial r_i}{\partial q_j} \right] \delta q_j$$

But we know that

$$\sum_{i=1}^{n} F_i^{(a)} \cdot \delta r_i = [Q_j]\, \delta q_j$$

where

$$Q_j = \sum_{i=1}^{n} F_i^{(a)} \cdot \frac{\partial r_i}{\partial q_j}$$

[are called the component of generalized force associated with the generalized coordinates q_j]

Now eq^n (5) becomes

$$\sum_{i=1}^{n} Q_j\, \delta q_j - \sum_{i=1}^{n} \dot{p}_i \cdot \delta r_i = 0 \quad \text{——(6)}$$

further, evaluating the II term of eq^n (6) we get

$$\sum_{i=1}^{n} \dot{p}_i \cdot \delta r_i = \sum_{i=1}^{n} m_i \overset{v_i}{\boxed{\ddot{r}_i}} \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j}\, \delta q_j$$

$$= \sum_{j=1}^{n} \left[\sum_{i=1}^{n} m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} \right] \delta q_j \quad \text{——(B)}$$

Now

$$\sum_{i=1}^{n} m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} = \left\{ \sum_{i=1}^{n} \frac{d}{dt}\left(m_i \dot{r}_i \cdot \frac{\partial r_i}{\partial q_j} \right) - m_i \dot{r}_i \cdot \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j} \right) \right] \quad \text{——(7)}$$

It is easy to prove that

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j} \right) = \frac{\partial}{\partial q_j}\left(\frac{d r_i}{dt} \right) = \frac{\partial v_i}{\partial q_j}$$

and

$$\frac{\partial r_i}{\partial q_j} = \frac{\partial v_i}{\partial \dot{q}_j}$$

then from eq^n (7)

$$\sum_{i=1}^{n} m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^{n} \frac{d}{dt}\left[m_i v_i \cdot \frac{\partial v_i}{\partial \dot{q}_j} \right] - m_i v_i \cdot \frac{\partial v_i}{\partial q_j}$$

substituting in eq^n (B)

$$\sum_{i=1}^{n} \dot{p}_i \cdot \delta r_i = \sum_{j=i=1}^{n} \left[\frac{d}{dt}\left(m_i v_i \cdot \frac{\partial v_i}{\partial \dot{q}_j} \right) - m_i v_i \cdot \frac{\partial v_i}{\partial q_j} \right] \delta q_j$$

$$= \sum_{j=1}^{n}\left[\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_j}\left(\sum_{i=1}^{n}\frac{1}{2}m_i(v_i \cdot v_i)\right)\right\} - \frac{\partial}{\partial q_j}\left\{\sum_{i=1}^{n}\frac{1}{2}m_i(v_i \cdot v_i)\right\}\right]\delta q_j$$

$$= \sum_{j=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\delta q_j \quad \text{---(8)}$$

where $T = \sum \frac{1}{2} m_i (\dot{r}_i)^2$

$T = \sum \frac{1}{2} m_i v_i^2$ (K.E. of the system)

Putting the value of (8) in (6)

$$\sum_{i=1}^{n} Q_j \cdot \delta q_j - \sum_{j=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\delta q_j = 0$$

$$\sum_{i=j=1}^{n}\left[Q_j - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_j}\right]\delta q_j = 0$$

$$\sum_{i=j=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j\right]\delta q_j = 0$$

Now constraints are holonomic & δq_j are independent of each other, then

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] = Q_j \quad \text{---(9)}$$

This eq^n ~~are~~ is valid in the case of conservative as well as non-conservative forces. These eq^ns are called Lagrange's eq^ns. However, this name is used mostly for eq^ns of the system when conservative forces are acting.

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For conservative system – w.k.t. if the system is conservative, force F_i are derivable from scalar potential energy fun^n V, Thus

$$F_i = -\nabla_i V = -\hat{i}\frac{\partial V}{\partial x_i} - \hat{j}\frac{\partial V}{\partial y_i} - \hat{k}\frac{\partial V}{\partial z_i} \quad \text{——(10)}$$

Now generalized force is given by

$$Q_j = \sum_{i=1}^{n} F_i \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^{n} (-\nabla_i V)\frac{\partial r_i}{\partial q_j} \quad \{\text{by eq}^n\ (10)$$

$$\therefore \quad Q_j = -\frac{\partial V}{\partial q_j} \quad \text{——(11)}$$

From eq^n (9) & (11)

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] = Q_j = -\frac{\partial V}{\partial q_j} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T-V) = 0$$

since the scalar potential function. V does not depends on $\dot{q}_j$ (Generalized velocity)

$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0 \quad \text{——(12)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \left(\frac{\partial L}{\partial q_j}\right) = 0 \quad \text{——(13)}$$

where $L = T - V$, which is called the Lagrangian of the system.

This eq^n are known as Langrange's eq^{n}s for Conservation system.

For Non Conservative system –

when the forces acting on the system consist of non conservative forces (f_i) in addition to the conservative forces (F_i), then the components of generalized forces can be written as (using eqn

$$Q_j = \sum_{i=1}^{n} [F_i^{(a)} + f_i] \cdot \frac{\partial r_i}{\partial q_j} \quad \text{—— (14)}$$

$$Q_j = \sum_{i=1}^{n} F_i^{(a)} \cdot \frac{\partial r_i}{\partial q_j} + \sum_{i=1}^{n} f_i \cdot \frac{\partial r_i}{\partial q_j}$$

$$Q_j = -\frac{\partial V}{\partial q_j} + Q_j' \quad \text{—— (15)}$$

where $Q_j' = \sum f_i \frac{\partial r_i}{\partial q_j}$ are the components of generalized non-potential force resulting from non-conservative forces

and $\sum f_i \frac{\partial r_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$ for conservative part

Here V is the scalar potential for conservative forces.

In such a case eqn (13) assumes the from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \left(\frac{\partial L}{\partial q_j}\right) = Q_j' \quad \text{—— (16)}$$

Eqn (16) represents the Lagrange's equations in the presence of non-conservative forces.