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Paper – II, Unit – I

Gauge Invariance of the Lagrangian →

If L is a Lagrangian for a system of n degrees of freedom, satisfying Lagrange's equations, it can be shown that

$$L' = L + \frac{dF}{dt} \quad \text{———} (1)$$

also satisfies Lagrange's equations, where F is an arbitrary function:-

$$F = F(q_1, q_2 \text{----} q_n, t) \quad \text{———} (2)$$

such a function is called Gauge function for the Lagrangian.

Proof:- Time derivative of the function F is

$$\frac{dF}{dt} = \sum_{k=1}^{n} \frac{\partial F}{\partial q_k} \dot{q}_k + \frac{\partial F}{\partial t} \quad \text{———} (3)$$

Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \quad \text{———} (4)$$

if $L' = L + \frac{dF}{dt}$ satisfies eqⁿ (4), then

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_k}\left(L + \frac{dF}{dt}\right)\right] - \frac{\partial}{\partial q_k}\left(L + \frac{dF}{dt}\right) = 0 \quad \text{———} (5)$$

subtracting eqⁿ (4) from (5), we get

$$\left[\left\{\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_k}\left(L + \frac{dF}{dt}\right)\right] - \frac{\partial}{\partial q_k}\left(L + \frac{dF}{dt}\right)\right\}\right] - \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k}\right] = 0$$

after solving

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_k}\left(\frac{dF}{dt}\right)\right] - \frac{\partial L}{\partial q_k}\left(\frac{dF}{dt}\right) = 0 \quad \text{———} (6)$$

if we prove L.H.S. of eqⁿ (6) to be equal to zero, then L' will satisfy the Lagrange's equations.

Now, L.H.S.

$$= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_k}\left(\frac{dF}{dt}\right) - \frac{\partial}{\partial q_k}\left(\frac{dF}{dt}\right)\right]$$

$$= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_k}\left(\frac{\partial F}{\partial t} + \sum_l \dot{q}_l \frac{\partial F}{\partial q_l}\right)\right] - \frac{\partial}{\partial q_k}\left[\frac{\partial F}{\partial t} + \sum_l \dot{q}_l \frac{\partial F}{\partial q_l}\right]$$

$$= \frac{d}{dt}\left[\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{q}_k}\right) + \sum_l \dot{q}_l \frac{\partial}{\partial q_l}\left(\frac{\partial F}{\partial \dot{q}_k}\right) + \sum_l \frac{\partial F}{\partial q_l}\frac{\partial \dot{q}_l}{\partial \dot{q}_k}\right]$$

$$- \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial t}\right) - \sum_l \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial q_l}\right)\dot{q}_l - \sum_l \frac{\partial F}{\partial q_l}\frac{\partial \dot{q}_l}{\partial q_k}$$

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Here $\frac{\partial F}{\partial \dot{q}_k} = 0$, $\frac{\partial \dot{q}_l}{\partial q_k} = 0$ and $\frac{\partial \dot{q}_l}{\partial \dot{q}_k} = \delta_{lk}$

Hence from the above eqn

$$= \frac{d}{dt}\left(\frac{\partial F}{\partial q_k}\right) - \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial t}\right) - \sum_l \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial q_l}\right)\dot{q}_l$$

$$= \frac{\partial}{\partial q_k}\left(\frac{dF}{dt}\right) - \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial t}\right) - \sum_l \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial q_l}\right)\dot{q}_l$$

$$= \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial t} + \sum_l \frac{\partial F}{\partial q_l}\dot{q}_l\right) - \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial t}\right) - \sum_l \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial q_l}\right)\dot{q}_l$$

$$= 0$$

Thus $L' = L + \frac{dF}{dt}$ satisfies the Lagrange's equations.