Nets

Topology II Semester
Unit IV

Reference book : K D Joshi
An Introduction to Topology, Wiley Eastern Limited, New Delhi

1. Definition. A directed set is a pair $(D, \geq)$ where D is a nonempty set and '$\geq$' is a relation on D satisfying
   (i) '$\geq$' is transitive ie whenever $m \geq n$ and $n \geq p$ then $m \geq p$ for all $m, n, p \in D$
   (ii) '$\geq$' is reflexive ie. $n \geq n$ for all $n \in D$.
   (iii) '$\geq$' directs the set D ie. for all $m, n \in D$, there exists $p \in D$ satisfying $p \geq m$ and $p \geq n$.

Examples
(i) $(\mathbb{N}, \geq)$ is a directed set where '$\geq$' is the usual ordering on $\mathbb{N}$.
(ii) A lattice is a directed set
(iii) $(\eta_x, \geq)$, ~~where~~ is a directed set where '$\geq$' means $U \geq V$ to mean $U \subset V$, here η_x denotes the neighbourhood system of a point x in a topological space X.

The condition (iii) for nets can be extended for finitely many elements of D ie. for $m_1, m_2, \ldots m_k \in D$, $\exists$ $p \in D$ satisfying $p \geq m_i$ for all $i = 1, 2, \ldots n$

Definition. A net in a ~~topology~~ set X is a function $S : D \longrightarrow X$, where D is a directed set.

Definition Let $(X, \mathcal{J})$ be a topological space and $S : D \to X$ be a net. Then the net S converges to a point $x \in X$ if given an open set U of X with $x \in U$, there exists $m \in D$ such that for all $n \in D$, $n \geq m$ implies that $S_n \in U$.

In this case we also say that x is a limit of the net S.

It is clear that the convergence of a net S depends upon the definition as well as the topology on the set X concerned.

Since the definition of nets is ~~directed~~ borrowed from the definition of sequences, a natural question arises about the uniqueness of limit of nets in a topological space X.

We have the following theorem answering this question

Theorem. A topological space is Hausdorff if and only if limits of all nets in it are unique.

Proof. Let $(X, \mathcal{J})$ be a Hausdorff space. Take a net $S : D \to X$ in X. If possible assume that the net S converges to x as well as to y in X.

We farther prove that $x=y$. To prove it, we proceed as below: assume that $x \neq y$. Since X is Hausdorff, there exist open sets U, V of X with

$$x \in U, \quad y \in V \quad \text{and} \quad U \cap V = \phi.$$

Next, we use the convergence of the net S.
Since the net S converges to x, $\exists\ m_1 \in D$ satisfying $S_n \in U$ for all $n \geq m_1$
Since the net S also converges to y, $\exists\ m_2 \in D$ satisfying $S_n \in U$ for all $n \geq m_2$.
Since $(D, \geq)$ is a directed set, $\exists\ m \in D$ satisfying $m \geq m_1$ and $m \geq m_2$.
Note that $S_m \in U$ and $S_m \in V$, combining the above observations.
It means that $U \cap V \neq \phi$. This contradiction gives that $x = y$.

Conversely, assume that in a topological space $(X, \mathfrak{J})$, limits of nets are unique. We show that X is a Hausdorff. To prove the assertion we use the method of contradiction again.

If possible assume that X is not a Hausdorff. Then there exists atleast one pair of points $x, y \in X$ with $x \neq y$ satisfying the condition that there does not exists disjoint neighbourhoods of x and y. Meaning thereby that if $x \in U$, $y \in V$, U, V open sets of X, then $U \cap V \neq \phi$.

We construct a net S in X by the procedure given below: Let $D = \eta_x \times \eta_y$.

Define a relation '$\geq$' in D as follows:
For $(U_1,V_1),(U_2,V_2) \in \eta_x \times \eta_y$
$(U_1,V_1) \geq (U_2,V_2)$ iff $U_1 \subset U_2$, $V_1 \subset V_2$
This makes $(D, \geq)$ a directed set. (Prove the conditions)

Define a net $S: D \to X$ as follows
$S(U,V) = z$, where z is any point of $U \cap V$. Since $U \cap V \neq \phi$, this definition is valid
Note that this definition is irrespective of the choice of the point z.

Finally we show that the net S converges to x: Let G be an open neighbourhood of x. Note that $(G, X) \in \eta_x \times \eta_y = D$. If $(U,V) \geq (G, X)$, then $U \subset G$ and so $S(U,V) \in U \cap V \subset U \subseteq G$. Hence the net S converges to the point x in X.

It can be similarly proved that the net S converges to $y \in X$. This completes the proof

Definition Let $(D, \geq)$ be a directed set and $E \subset D$. Then E is called an eventual subset of D if there exists $m \in D$ such that whenever $n \in D$ with $n \geq m$, then $n \in E$.

A net $S: D \to X$ is said to be eventually in a subset A of X if the set $S^{-1}(A)$ is an eventual subset of D.

Definition. Let $(D, \geq)$ be a directed set. A subset F of D is said to be a cofinal subset of D if for every $m \in D$, there exists $n \in F$ such that $n \geq m$. A net $S : D \to X$ is said to be frequently in a subset A of X if $S^{-1}(A)$ is a cofinal subset of D.

Note that every eventual subset is a cofinal subset.

A cofinal subset need not be eventual: For example in N (natural numbers), any infinite set is cofinal but not necessarily eventual.

Consider the usual ordering in η_x, the neighbourhood system at x in a topological space $(X, \mathcal{T})$. Then $\mathcal{L}_x$, the local base at x is an important example of a cofinal subset.

A net $S : D \to X$ is said to be frequently in a subset A of X if $S^{-1}(A)$ is a cofinal subset of D.

Let $S : D \to X$ be a net in X. A point $x \in X$ is said to be a cluster point of S if for every neighbourhood U of x in X and $m \in D$, there exists $n \in D$ such that $n \geq m$ and $S_n \in U$.