Method of Partial Waves —

This method has been found to be useful in solving the scattering problems & collision problems. Further this method has been found to be applicable mainly to spherical symmetrical potential & involves the expansion of the wave function as a series of spherical harmonics multiplied by a radial wave funct^n. The method is useful when the energy of incident particle is low i.e. it is the method of low energy scattering.

For spherically symmetric potential the angular momentum of the scattered particle is a const. of motion. It is therefore advantageous to develop solu. in term of angular momentum eigen funct^n. If the energy of incident particle is low, it will turn out that only few eigen funct^n with small angular momenta will be really affected by the potential.

Assume that incident momentum is $\hbar k$. We assume that particles of momentum $\hbar k$ are incident along Z-direct^n.

The free particle Schrodinger eq. is

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(r) = E\psi(r)$$

$$= \frac{\hbar^2 k^2}{2\mu}\psi(r) \quad \text{--- (1)} \qquad \left\{ \because E = P^2/2m, \; P = \hbar k \right.$$

$$\therefore -\frac{\hbar^2}{2\mu}\nabla^2\psi(r) = \frac{\hbar^2 k^2}{2\mu}\psi(r)$$

or

$$[-\nabla^2 - k^2]\,\psi(r) = 0 \quad \text{--- (2)}$$

The solu. is

$$\psi(r) = A.\,e^{i\vec{k}.\vec{r}}$$

The plane wave solu. representing the incident particle can be expanded in term of angular momentum eigen funct? i.e. Spherically Bessel's funct? (j_ℓ)

$$\exp\{i k \cdot r\} = \sum_{\ell=0}^{\infty} i^\ell \left[(2\ell+1)\, P_\ell(\cos\theta)\right] j_\ell(kr) \quad \text{---} \ (3)$$

For large 'r' the asymptotic form of j_ℓ is

$$j_\ell(kr) \xrightarrow[r\to\infty]{} \frac{\sin(kr - \ell\pi/2)}{kr}$$

$$\xrightarrow[r\to\infty]{} \frac{\exp[i(kr-\ell\pi/2)] - \exp[-i(kr-\ell\pi/2)]}{2ikr} \quad \text{---} \ (4)$$

With the insertion of the factor $e^{-i\omega t}$, it is clear from eq. (3) & (4) that the plane wave contains equal amplitudes of incoming & outgoing spherical waves.
Thus, the asymptotic form of eq. (3) will be

$$e^{ikr} = e^{ikr\cos\theta} \quad \text{---} \ (5)$$

$$\rightarrow \sum_\ell \frac{(2\ell+1)}{2ikr}\left[e^{ikr} - (-1)^\ell e^{-ikr}\right] P_\ell \cos\theta \ \text{---} \ (6)$$

(Central Potential) –
The Schrodinger eq. for the problem is:

$$\left[\nabla^2 + k^2\right]\psi(r) = U\psi(r) \quad \text{---} \ (7)$$

We now expand $\psi(r)$ in the form

$$\psi(r) = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{k}\, i^\ell\, \frac{u_\ell(r)}{r}\, P_\ell(\cos\theta) \quad \text{---} \ (8)$$

where $\frac{u_\ell(r)}{r}$ is the radial part of the sol.

$P_\ell(\cos\theta)$ is angular dependent. & $\frac{(2\ell+1)}{k} i^\ell$ has been introduced for convinience

For, eq. ⑧ to be correct, $u_\ell(r)$ must satisfy the Schrodinger eq.

$$\frac{d^2u_\ell}{dr^2} + \left[k^2 - U(r) - \frac{\ell(\ell+1)}{r^2}\right]u_\ell(r) = 0 \quad \text{— (9)}$$

For a short-range force if

$$\lim_{r\to\infty} r\,V(r) = 0 \quad \text{— (10)}$$

The asymptotic form of $u_\ell(r)$ will be $\exp(\pm ikr)$; thus we may write.

$$u_\ell(r) \xrightarrow[r\to\infty]{} A_\ell e^{ikr} + B_\ell e^{-ikr}$$

$$= C_\ell \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right) \quad \text{— (11)}$$

Where the factor $\frac{\ell\pi}{2}$ has been introduced for the sake of convenience & 'δ_ℓ' are known as phase shift.

eq. (11) can be written as

$$u_\ell \simeq \frac{1}{2i} C_\ell (-1)^\ell \left[e^{i\delta_\ell} e^{ikr} - (-1)^\ell e^{-i\delta_\ell} e^{-ikr}\right] \quad \text{— (12)}$$

Substituting (12) in (8) we obtained

$$\psi(r) \xrightarrow[r\to\infty]{} \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2ikr} C_\ell \left[e^{i\delta_\ell} e^{ikr} - (-1)^\ell e^{-i\delta_\ell} e^{-ikr}\right] P_\ell(\cos\theta) \quad \text{— (13)}$$

eq. (13) is to be compared with the form

$$\psi(r) \to e^{ikr} + \frac{e^{ikr}}{r} f(\theta)$$

$$\therefore \psi(r) = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{2ikr} P_\ell(\cos\theta)\left[e^{ikr} - (-1)^\ell e^{-ikr}\right] + \frac{e^{ikr}}{r} f(\theta) \quad \text{— (14)}$$

eq. (13) & (14) represent the same funct^n. Hence the coffi. of $\exp(ikr)$ & $\exp(-ikr)$ should be identical. Equatting the coffi. of exp(−ikr), gives

$$C_l = e^{i\delta_l}$$

& also from coffi. of exp(ikr)

$$f(\theta) = \sum_{l=0}^{\infty} \frac{(2l+1)}{2ik} \{ \exp(2i\delta_l) - 1 \} P_l(\cos\theta) \quad \text{---(15)}$$

or

$$f(\theta) = \sum_{l=0}^{\infty} \frac{(2l+1)}{k} \sin\delta_l \exp(i\delta_l) \cdot P_l(\cos\theta) \quad \text{---(16)}$$

∴ The total cross-sectⁿ can be shown to be

$$\sigma = \int |f(\theta)|^2 d\omega$$

$$\boxed{\sigma = \frac{4\pi}{K^2} \sum_{l=0}^{\infty} (2l+1) \sin^2\delta_l}$$

The total cross-sectⁿ σ will be equal to the total cross-sectⁿ upto the l^{th} partial wave where δ_l represents the phase shift of l^{th} partial wave. The expression follows that as far as the total cross-sectⁿ is concerned, the different partial waves contribute independently, there being no interference.