→ Time dependent perturbation theory :-

Qu. Discuss the essential of time dependent perturbation theory. Show that for constant 'P' the probability of reaching to state 's' from initial state 'n' is given by.

$$\rightarrow \quad P_{n\to s} = \frac{|V_{sn}|^2}{\hbar^2} \frac{\sin^2 \frac{\omega_{sn} t}{2}}{\left(\frac{1}{2}\omega_{sn}\right)^2}$$

Solu:- Let us consider a system for which the schrodinger time dependent equation is given by,

$$\rightarrow \quad H\psi = i\hbar \frac{\partial \psi}{\partial t}$$

which can also be represented as,

$$\rightarrow \quad H|\psi_n(t)\rangle = i\hbar \frac{\partial}{\partial t} |\psi_n(t)\rangle \quad \text{——— (1)}$$

$$\rightarrow \quad \psi_n(t) = \sum_n C_n |n\rangle e^{-i\omega_n t} \quad \text{——— (2)}$$

where, $\omega_n = E_n/\hbar$

and $C_n = C_n(t)$

now consider,

$$\rightarrow \quad H = \underbrace{H_0}_{\text{unpurturbed part}} + \underbrace{V}_{\text{perturbad part}} \quad \text{——— (3)}$$

putting the value of equation 2 & 3 in 1 we get,

$$\rightarrow \quad (H_0+V)\left[\sum_n C_n|n\rangle e^{-i\omega_n t}\right] = i\hbar\frac{\partial}{\partial t}\left[\sum_n C_n|n\rangle e^{-i\omega_n t}\right]$$

$$\rightarrow \quad (E_n+V)\left[\sum C_n|n\rangle e^{-i\omega_n t}\right] = i\hbar\left[\sum C_n|n\rangle e^{-i\omega_n t}(-i\omega_n) + e^{-i\omega_n t}\sum \dot{C}_n|n\rangle\right]$$

$$\left\{\begin{array}{ll}\text{where,} & H_0 = E_n \\ \text{and} & \dot{C}_n = \frac{d}{dt}C_n\end{array}\right\}$$

$$\rightarrow \quad E_n \sum \cancel{C_n|n\rangle e^{-i\omega_n t}} + V\sum C_n|n\rangle e^{-i\omega_n t} = \cancel{i\hbar}\sum C_n|n\rangle \cancel{e^{-i\omega_n t}}\left(-\cancel{i}\frac{E_n}{\cancel{\hbar}}\right) + e^{-i\omega_n t}\sum \dot{C}_n|n\rangle\, i\hbar$$

$$\rightarrow \quad V\sum C_n|n\rangle e^{-i\omega_n t} = i\hbar\sum \dot{C}_n|n\rangle e^{-i\omega_n t} \qquad (4)$$

multiply both side by $\langle s|$.

$$\rightarrow \quad \langle s|V\sum C_n|n\rangle e^{-i\omega_n t} = i\hbar\sum_n \dot{C}_n\langle s|n\rangle e^{-i\omega_n t}$$

but we know that,

$$\langle s|n\rangle = \begin{cases}\delta_{sn}=1, & \text{if } s=n \\ \delta_{sn}=0, & \text{if } s\neq n.\end{cases}$$

now, using the Orthonormality condition,

$$\rightarrow \sum_n \langle s|V|n\rangle C_n e^{-i\omega_n t} = i\hbar \sum_s \dot{C}_s e^{-i\omega_s t}$$

$$\rightarrow \dot{C}_s = \frac{1}{i\hbar} \sum \langle s|V|n\rangle C_n e^{i(\omega_s - \omega_n)t}$$

$$\rightarrow \dot{C}_s = \frac{1}{i\hbar} \sum C_n(t) e^{i\omega_{sn}t} V_{sn} \quad \text{(5)}$$

$$\left\{ \begin{array}{l} \because C_n = C_n(t) \text{ and } V_{sn} = \langle s|V|n\rangle \\ \text{also, } \omega_{sn} = (\omega_s - \omega_n) \end{array} \right\}$$

now, when the perturbation are small the R.H.S. of equation (5) we can replace $C_n(t)$ by $C_n(0)$ and for suppose that the system initially at the state 'n' then,

$$\rightarrow C_n(0) = 1$$

now, under such condition,

$$\rightarrow C_n(t) = C_n(0) = 1$$

i.e.

$$\rightarrow \dot{C}_s = \frac{1}{i\hbar} e^{i\omega_{sn}t} V_{sn}$$

now, on integrating the equation over 't', we get,

$$\rightarrow \boxed{C_s(t) = \frac{1}{i\hbar} \int_0^t V_{sn} e^{i\omega_{sn}t} dt} \quad \text{(6)}$$

From equation (6) we get amplitude.

now, when the perturbation is constant,

$$\rightarrow \quad C_s(t) = \frac{V_{sn}}{i\hbar} \int_0^t e^{i\omega_{sn}t}\, dt$$

$$\rightarrow \quad C_s(t) = \frac{V_{sn}}{i\hbar} \left[\frac{e^{i\omega_{sn}t}}{i\omega_{sn}} \right]_0^t$$

$$= \frac{V_{sn}}{i^2 \hbar \omega_{sn}} \left[e^{i\omega_{sn}t} - e^0 \right]$$

$$= \frac{V_{sn}}{i^2 \hbar \omega_{sn}} \left[e^{i\omega_{sn}t} - 1 \right]$$

$$= \frac{V_{sn}}{i^2 \hbar \omega_{sn}} \left[e^{\frac{i\omega_{sn}t}{2}} \cdot e^{\frac{i\omega_{sn}t}{2}} - 1 \right]$$

$$= \frac{V_{sn}}{i^2 \hbar \omega_{sn}}\, e^{i\omega_{sn}t/2} \left[e^{i\omega_{sn}t/2} - e^{-i\omega_{sn}t/2} \right]$$

$$= \frac{V_{sn}}{i^2 \hbar \omega_{sn}}\, e^{i\omega_{sn}t/2}\, 2i \sin(\omega_{sn}t/2)$$

then, the probability (transition probability) for reaching to state 'S' is,

$$\longrightarrow \quad P_{n\to s}(t) = |C_s(t)|^2$$

and finally we have;

$$\longrightarrow \quad \boxed{P_{n\to s} = 4 \cdot \frac{|V_{sn}|^2}{\hbar^2} \frac{\sin^2(\omega_{sn} t/2)}{(\omega_{sn})^2}}$$

Hence proved.