Adiabatic & Sudden approximation explanation

Here we study approximation method that involve the rate of change of the Hamiltonian, rather than the magnitude of the time dependent part of the Hamiltonian.

If the Hamiltonian changes very slowly with the time, we expect to be able to approximate solution of schrodinger equation by means of stationary energy eigen functions of the instantaneous Hamiltonian, so that a particular eigenfunction at one time goes over continuously into the corresponding eigen function at a later time (Adiabatic approximation).

If the Hamiltonian changes from one steady form to another over a very short time interval, we expect that the wave function does not change much, although the expansion of this function in eigenfunctions of the initial and final Hamiltonians may be quite different (Sudden approximation).

Sudden approximation :-

If the perturbing potentials undergo suddenly a large change and then become constants, under these circumstances the Hamiltonian undergoes a large change in a small but finite time.

Suppose Hamiltonian changes from H_0 to H_1 suddenly after time $t=0$, the wave fun. of this is $\psi_n e^{-\frac{iE_n^0 t}{\hbar}}$ and the

equation can be written as $H_0\psi_n = E_n^0\psi_n$ and after the perturbation the wave function becomes, $\psi_m e^{-\frac{iE_m t}{\hbar}}$ and the

wave equation can be written as $H_1\psi_m = E_m\psi_m$

We know that the perturbed wave function can be expressed as a linear combination of unperturbed wave function as

$$\rightarrow \psi_m(x) = \sum_n C_{mn}\psi_n(x) \quad \text{—— (A)}$$

Now, the coff. of C_{mn} can be evaluated by

$$\rightarrow C_{mn} = \frac{\int \psi_n^*(x)\,\psi_m(x)\,d\tau}{\int \psi_n^*(x)\,\psi_n(x)\,d\tau} = 1$$

$$= \int \psi_n^*(x)\,\psi_m(x)\,d\tau$$

i.e. eq. (A) can be written as

$$\rightarrow \boxed{\psi_m(x) = \sum_n \left\{\int \psi_n^*(x)\,\psi_m(x)\,d\tau\right\}\psi_n(x)\,e^{-\frac{iE_n^0 t}{\hbar}}}$$

It can be seen from above eq. that wave fun. does not change while Hamiltonian is changing.

Adiabatic approximation :-

In the adiabatic case, we expect that solution of the schrodinger equation can be approximated by means of stationary eigen functions of the instantaneous Hamiltonian, so that a particular eigen function at one time goes over continuously into the corresponding eigen function at a later time.
then,

$$\rightarrow \quad H(t)\, U_n(t) = E_n(t)\, U_n(t) \quad \text{—— (1)}$$

can be solved at each instant of time, we expect that a system that is in a discrete nondegenerate state $U_m(0)$ with energy $E_m(0)$ at $t=0$ is likely to be in the state $U_m(t)$ with energy $E_m(t)$ at time t, provided that $H(t)$ changes very slowly with time.

Our objective is to estimate the extant to which this expectation is not fulfilled, so that other states appear in a expansion of ψ in term of U's.

The wave function ψ satisfies the time-dependent schrodinger equation,

$$\rightarrow \quad i\hbar \frac{\partial \psi}{\partial t} = H(t)\,\psi. \quad \text{—— (2)}$$

we proceed by expanding ψ in terms of the U's in the following way.

$$\rightarrow \quad \psi = \sum_n a_n(t)\, U_n(t) \exp\left[(i\hbar)^{-1} \int_0^t E_n(t')\, dt'\right] \quad \text{—— (3)}$$

Were, we assume that the U_n are orthonormal, discrete, and nondegenerate. substituting eq. 3 in eq. 2 gives,

$$\rightarrow \sum_n \left[\dot{a}_n U_n + a_n \frac{\partial U_n}{\partial t} \right] \exp\left[(i\hbar)^{-1} \int_0^t E_n(t')\, dt' \right] = 0$$

(by eq. 1) also

now multiply by U_k^* and integrate over all space to obtain.

$$\rightarrow \dot{a}_k = -\sum_n a_n \langle k|\dot{n}\rangle \exp\left[(i\hbar)^{-1} \int_0^t (E_n \text{[redacted]} E_k)\, dt' \right]$$

$$\because \langle k|\dot{n}\rangle = \int U_k^* \frac{\partial U_n}{\partial t} d^3r$$

an expansion for $\langle k|\dot{n}\rangle$ can be found by differentiating eq. 1 w.r.t. time t.

$$\rightarrow \frac{\partial H}{\partial t} U_n + H \frac{\partial U_n}{\partial t} = \frac{\partial E_n}{\partial t} U_n + E_n \frac{\partial U_n}{\partial t}$$

now we multiply on left by U_k^*, where $k \neq n$, integrate over all space, and make use of eq. 1 to obtain,

$$\rightarrow \langle k| \frac{\partial H}{\partial t} |n\rangle = (E_n - E_k) \langle k|\dot{n}\rangle \qquad k \neq n$$

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