Unit-I

Some Preliminary Formulae

§ 1.1-1. (A) DIFFERENTIAL COEFFICIENTS OF SOME ELEMENTARY FUNCTIONS

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$

(ii) $\frac{d}{dx}(e^x) = e^x$

(iii) $\frac{d}{dx}(a^x) = a^x \log_e a$

(iv) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(v) $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$

(vi) $\frac{d}{dx}(\sin x) = \cos x$

(vii) $\frac{d}{dx}(\cos x) = -\sin x$

(viii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(ix) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(x) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(xi) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(xii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(xiii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

(xiv) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(xv) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(xvi) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

(xvii) $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$.

§ 1.1-1. (B) HYPERBOLIC FUNCTIONS

The hyperbolic functions are defined as follows :

(i) $\cosh x = \frac{1}{2}(e^x + e^{-x})$

(ii) $\sinh x = \frac{1}{2}(e^x - e^{-x})$

(iii) $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

(iv) $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

(v) $\operatorname{cosech} x = \frac{2}{e^x - e^{-x}}$

(vi) $\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$.

Some Important Results :

(i) $\cosh^2 x - \sinh^2 x = 1$

(ii) $\cosh^2 x + \sinh^2 x = \cosh 2x$

(iii) $\cosh 2x = 2\cosh^2 x - 1$

(iv) $\cosh 2x = 2\sinh^2 x + 1$

(v) $\sinh 2x = 2 \sinh x \cosh x$

(vi) $\sinh^{-1} x = \log\{x + \sqrt{x^2+1}\}$

(vii) $\cosh^{-1} x = \log\{x + \sqrt{x^2-1}\}$.

Differential Coefficients :

(i) $\frac{d}{dx}(\sinh x) = \cosh x$

(ii) $\frac{d}{dx}(\cosh x) = \sinh x$

(iii) $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

(iv) $\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$

(v) $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

(vi) $\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$.

§ 1.1-1. (C) SOME FORMULAE OF INVERSE CIRCULAR FUNCTIONS

(i) $\tan^{-1}\left(\frac{x+y}{1-xy}\right)=\tan^{-1}x+\tan^{-1}y$

(ii) $\tan^{-1}\left(\frac{x-y}{1+xy}\right)=\tan^{-1}x-\tan^{-1}y$

(iii) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)=2\tan^{-1}x$

(iv) $\sin^{-1}x\pm\sin^{-1}y=\sin^{-1}\{x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\}$

(v) $\cos^{-1}x\pm\cos^{-1}y=\cos^{-1}\{xy\mp\sqrt{1-x^2}\sqrt{1-y^2}\}$

(vi) $\tan^{-1}x=\cot^{-1}(1/x)$

(vii) $\cot^{-1}x=\tan^{-1}(1/x)$

(viii) $\sin^{-1}x+\cos^{-1}x=\frac{1}{2}\pi$

(ix) $\tan^{-1}x+\cot^{-1}x=\frac{1}{2}\pi$

(x) $\sec^{-1}x+\text{cosec}^{-1}x=\frac{1}{2}\pi$

(xi) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)=2\tan^{-1}x$

(xii) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=2\tan^{-1}x.$

§ 1.1-2. SUCCESSIVE DIFFERENTIATION OF ANY FUNCTION

If y [or $f(x)$] be a differentiable function of x then $\frac{dy}{dx}$ or y' or Dy or y_1 or $f'(x)$ is called the first derivative or first **differential coefficient** of the function y [or $f(x)$] with respect to x. The differential coefficient may be a constant or a function of x. If the first derivative is differentiated again, then the derivative so obtained is called **second derivative** of y [or $f(x)$] and is denoted by

$$\frac{d}{dx}\left(\frac{dy}{dx}\right),\ \frac{d^2y}{dx^2},\ y'',\ D^2y,\ y_2 \text{ or } f''(x).$$

Similarly third, fourth..., nth derivatives of y are denoted respectively by

$$\frac{d^3y}{dx^3}\ \{\text{or } y''' \text{ or } D^3y, \text{ or } y_3 \text{ or } f'''(x)\},$$

$$\frac{d^4y}{dx^4}\ \{\text{or } y'''' \text{ or } D^4y, \text{ or } y_4 \text{ or } f''''(x)\}, \ldots,$$

$$\frac{d^ny}{dx^n}\ \{\text{or } y^{(n)} \text{ or } D^ny, \text{ or } y_n \text{ or } f^{(n)}(x)\}.$$

§ 1.1-3. *n*th DERIVATIVE OF SOME STANDARD FUNCTIONS

(i) If $y=e^{ax+b}$, then

$$y_1=ae^{ax+b},\ y_2=a^2e^{ax+b}, \ldots \text{ etc.}$$

In general

$$y_n=a^ne^{ax+b}$$

$$D^n(e^{ax+b})=a^n e^{ax+b}.$$

Particular case 1. If $a=1, b=0$ then $D^ne^x=e^x$.

Particular case 2. If $y=a^x=e^{x\log a}$ then

$$y_n=(\log a)^n e^{x\log a}=(\log a)^n a^x.$$

$$D^na^x=(\log a)^n a^x.$$

(ii) $y=(ax+b)^m$, then

$$y_1=m(ax+b)^{m-1}a$$

$$y_2=m(m-1)(ax+b)^{m-2}a^2$$

$$y_3=m(m-1)(m-2)(ax+b)^{m-3}a^3, \text{ etc.}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$y_n=m(m-1)(m-2)\ldots(m-n+1)(ax+b)^{m-n}a^n.$$

$$\mathbf{D^n(ax+b)^m=m(m-1)(m-2)\ldots(m-n+1)\,a^n\,(ax+b)^{m-n}} \quad \textbf{if } n<m.$$

Particular cases 1. If m is a positive integer then

$$D^n(ax+b)^m=\frac{m(m-1)(m-2)\ldots(m-n+1)(m-n)\ldots 2.1}{(m-n)\ldots 2.1}\cdot a^n(ax+b)^{m-n}$$

$$=\frac{m!}{(m-n)!}a^n(ax+b)^{m-n}.$$

If $m<n$, then

$$\mathbf{D^n(ax+b)^m=0.}$$

2. If m is a positive integer and $m=n$, then

$$D^n(ax+b)^n=\frac{n!}{0!}a^n(ax+b)^0=n!\,a^n.$$

If $a=1, b=0$, then

$$D^nx^n=n!.$$

3. If m is a negative integer then let $m=-p$ where p is a + ve integer, then

$$D^n(ax+b)^{-p}=(-p)(-p-1)\ldots\{-p-(n-1)\}\,a^n(ax+b)^{-p-n}$$

$$=(-1)^n p(p+1)\ldots(p+n-1)\,a^n(ax+b)^{-p-n}$$

$$=\frac{(-1)^n(p+n-1)!}{(p-1)!}\cdot a^n(ax+b)^{-p-n}.$$

4. If $m=-1$ *i.e.*, $p=1$ then

$$D^n(ax+b)^{-1}=(-1)^n\,n!\,a^n(ax+b)^{-n-1}.$$

(iii) If

$$y=\log(ax+b),$$

$$y_1=\frac{a}{ax+b}=a(ax+b)^{-1}$$

$$y_2=(-1)a^2(ax+b)^{-2}$$

$$y_3=(-1)(-2)a^3(ax+b)^{-3}, \text{ etc.}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$y_n=(-1)(-2)(-3)\ldots\{-(n-1)\}\,a^n(ax+b)^{-n}$$

$$=(-1)^{n-1}(n-1)!\,a^n(ax+b)^{-n}.$$

$$\therefore \quad \mathbf{D^n\log(ax+b)=\frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}}.$$

(iv) If $y=\sin(ax+b)$, then

$$y_1=a\cos(ax+b)=a\sin(ax+b+\tfrac{1}{2}\pi)$$

$$y_2=a^2\cos(ax+b+\tfrac{1}{2}\pi)=a^2\sin(ax+b+\tfrac{1}{2}\cdot 2\pi)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$y_n=a^n\sin(ax+b+\tfrac{1}{2}\cdot n\pi)$$

$$\mathbf{D^n\sin(ax+b)=a^n\sin(ax+b+\tfrac{1}{2}n\pi).}$$

Similarly,

$$\mathbf{D^n\cos(ax+b)=a^n\cos(ax+b+\tfrac{1}{2}\cdot n\pi).}$$

(v) If $y = e^{ax} \cos (bx + c)$, then

$$y_1 = e^{ax} . a \cos (bx + c) - e^{ax} . b \sin (bx + c) = e^{ax} [a \cos (bx + c) - b \sin (bx + c)].$$

Let $a = r \cos \phi,\ b = r \sin \phi.$ ...(1)

$r^2 = a^2 + b^2,\ \phi = \tan^{-1} (b/a).$

Thus $y_1 = e^{ax} [r \cos \phi \cos (bx + c) - r \sin \phi \sin (bx + c)] = r . e^{ax} \cos (bx + c + \phi).$

$\therefore$ $y_2 = r^2 e^{ax} \cos (bx + c + 2\phi)$, etc.

Similarly, $y_n = r^n e^{ax} \cos (bx + c + n\phi).$

In general $D^n \{e^{ax} \cos (bx + c)\} = r^n e^{ax} \cos (bx + c + n\phi).$

$\therefore$ Similarly, proceeding as above, we have

$$D^n \{e^{ax} \sin (bx + c)\} = r^n e^{ax} \sin (bx + c + n\phi)$$

where r and ϕ are given by equation (1).

Ex. *Find the nth differential coefficient of the following :*

(i) $\tan^{-1} (x/a)$. (ii) $\tan^{-1} x$.

(iii) $\tan^{-1} \left\{\frac{1+x}{1-x}\right\}$. (iv) $\tan^{-1} \left\{\frac{2x}{1-x^2}\right\}$.

Sol. (i) $y = \tan^{-1} (x/a)$, then

$$y_1 = \frac{1}{1 + (x^2/a^2)} \cdot \frac{1}{a} = \frac{a}{x^2 + a^2}$$

$$y_1 = \frac{a}{(x + ai)(x - ai)} = \frac{a}{(-ia - ia)(x + ai)} + \frac{a}{(ai + ai)(x - ai)}$$

$$= \frac{1}{2i}\left[\frac{1}{(x - ai)} - \frac{1}{(x + ai)}\right] = \frac{1}{2i} [(x - ai)^{-1} - (x + ai)^{-1}].$$

Now differentiating both sides $(n - 1)$ times, we have

$$y_n = \frac{(-1)^{n-1} (n-1)!}{2i} [(x - ai)^{-n} - (x + ai)^{-n}] \quad ...(1)$$

Put $x = r \cos \phi$ and $a = r \sin \phi$

$$(x - ai)^{-n} = r^{-n} (\cos \phi - i \sin \phi)^{-n} = r^{-n} (\cos n\phi + i \sin n\phi).$$

Similarly, $(x + ai)^{-n} = r^{-n} (\cos n\phi - i \sin n\phi).$

$\therefore$ $(x - ai)^{-n} - (x + ai)^{-n} = r^{-n} (2i \sin n\phi).$

Substituting values in equation (1), we get

$$y_n = (-1)^{n-1} (n-1)!\, r^{-n} \sin n\phi$$

$$= (-1)^{n-1} (n-1)!\, (a/\sin \phi)^{-n} \sin n\phi \quad [\because r = a/\sin \phi]$$

$$= (-1)^{n-1} (n-1)!\, a^{-n} \sin^n \phi \sin n\phi, \text{ where } \phi = \tan^{-1} (a/x).$$

(ii) $y = \tan^{-1} x.$

Proceeding as (i) above or put $a = 1$ in above result, we have

$$y_n = (-1)^{n-1} (n-1)!\, \sin^n \phi \sin n\phi, \text{ where } \phi = \tan^{-1} (1/x).$$

(iii) $y = \tan^{-1} \left\{\frac{1+x}{1-x}\right\} = \tan^{-1} \left\{\frac{1+x}{1-1.x}\right\} = \tan^{-1} 1 + \tan^{-1} x.$

$\therefore$ $y_1 = \frac{1}{1+x^2}.$

Now proceeding as (i) above when $a = 1$, we get

$$y_n = (-1)^{n-1} (n-1)!\, \sin^n \phi \sin n\phi, \text{ where } \phi = \tan^{-1} (1/x).$$

(iv) $y = \tan^{-1} \left\{\frac{2x}{1-x^2}\right\} = 2 \tan^{-1} x$, therefore from (i),

$$y_n = 2(-1)^{n-1} (n-1)!\, \sin^n \phi \sin n\phi, \text{ where } \phi = \tan^{-1} (1/x).$$

§ 1.1-4. LEIBNITZ'S THEOREM

The nth differential coefficient of the product of two functions is conveniently evaluated by the use of this theorem. Its statement is :

Theorem. *If u and v are two functions of x, then*

$$D^n (uv) = D^n u . v + {}^nC_1 D^{n-1} u\, Dv + {}^nC_2 D^{n-2} u\, D^2 v + ... + {}^nC_r D^{n-r} u\, D^r v + ... + u\, D^n v.$$

Important Note I. While applying Leibnitz's theorem if one of the two functions is such that its higher differential coefficients become zero then this function should be taken as v and the remaining function u.

II. The formula of Leibnitz's theorem can be rewritten in the following form, [by taking successive integration of $D^n (u)$ and successive differentiation of v]

$$D^n (u . v) = D^n u . v + {}^nC_1 \{\int (D^n u)\, dx\}\, Dv + {}^nC_2 [\int \{\int (D^n u)\, dx\}\, dx]\, D^2 v + ... + u . D^n v.$$

Ex. *If $y = x^2 e^x$ then prove that $y_n = \frac{1}{2} n(n-1) \frac{d^2y}{dx^2} - n(n-2) \frac{dy}{dx} + \frac{1}{2} (n-1)(n-2)\, y.$*

Sol. $y = x^2 e^x$ ...(1)

$\therefore$ $\frac{dy}{dx} = x^2 e^x + 2x e^x = y + 2x e^x$ ...(2)

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + 2 [x e^x + e^x] = \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) + 2e^x \quad \text{[From (1)]}$$

$$= \frac{2dy}{dx} - y + 2e^x. \quad ...(3)$$

Now $D^n (x^2 e^x) = y_n = D^n (e^x) . x^2 + {}^nC_1 D^{n-1} (e^x) . 2x + {}^nC_2 D^{n-2} (e^x) . 2$

$$= e^x . x^2 + 2nx\, e^x + n(n-1)\, e^x$$

$$= y + n\left(\frac{dy}{dx} - y\right) + n(n-1) \times \frac{1}{2}\left(\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y\right) \quad \text{[From (1), (2) and (3)]}$$

$$= \frac{1}{2} n(n-1) \frac{d^2y}{dx^2} - n(n-2) \frac{dy}{dx} + \frac{1}{2} (n-1)(n-2)\, y.$$

Chapter 2

Expansion of Functions by Maclaurin's and Taylor's Theorem

§ 1.2-1. MACLAURIN'S THEOREM

If $f(x)$ be a function of the variable x such that it can be expanded in ascending powers of x and this expansion be differentiable any number of times, then the theorem states that

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^n}{n!}f^{(n)}(0)+\ldots \qquad \ldots(1)$$

Proof. Let $\quad f(x)=A_0+A_1x+A_2x^2+A_3x^3+\ldots$

where $A_0, A_1, A_2, \ldots$ are constants.

Now by successive differentiation of (1), w.r.t. x, we have

$$f'(x)=A_1 . 1+2A_2x+3A_3x^2+4A_4x^3+\ldots$$

$$f''(x)=2.1A_2+3.2A_3x+4.3A_4x^2+\ldots$$

$$f'''(x)=3.2.1A_3+4.3.2A_4x+\ldots \text{ etc.}$$

Substituting $x=0$ in each of above relations, we get

$$f(0)=A_0, f'(0)=A_1, f''(0)=2!\,A_2, f'''(0)=3!\,A_3 \ldots f^{(n)}(0)=n!\,A_n \text{ etc.}$$

$\therefore \quad A_0=f(0), A_1=f'(0), A_2=f''(0)/2!, A_3=f'''(0)/3! \ldots, A_n=f^n(0)/n!$ etc.

Substituting the values of $A_0, A_1, A_2, \ldots$ in eqn. (1), we have

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^n}{n!}f^{(n)}(0)+\ldots \qquad \ldots(2)$$

Note. If $f(x)$ be denoted by 'y' then (2) may be written as :

$$y=(y)_0+\frac{x}{1!}(y_1)_0+\frac{x^2}{2!}(y_2)_0+\ldots+\frac{x^n}{n!}(y_n)_0+\ldots \qquad \ldots(3)$$

§ 1.2-2. TAYLOR'S THEOREM

If $f(a+h)$, [where a is independent of h] be a function of the variable h such that it can be expanded in ascending powers of h and this expansion be differentiable any number of times then the theorem states that

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^n}{n!}f^n(a)+\ldots$$

Proof. Let $\quad f(a+h)=A_0+A_1h+A_2h^2+A_3h^3+\ldots+A_nh^n+\ldots \qquad \ldots(1)$

where $A_0, A_1, A_2, \ldots, A_n$ are constants independent of h.

Now by successive differentiation of (1) w.r.t. 'h', we have

$$f'(a+h)=A_1+2A_2h+3A_3h^2+4A_4h^3+\ldots$$

$$f''(a+h)=2.1A_2+3.2A_3h+4.3A_4h^2+\ldots$$

$$f'''(a+h)=3.2.1A_3+4.3.2A_4h+\ldots \text{ etc.}$$

Putting $h=0$, in each of above relations, we have

$$f(a)=A_0, f'(a)=A_1, f''(a)=2!\,A_2, f'''(a)=3!\,A_3, \ldots, f^n(a)=n!\,A_n, \text{ etc.}$$

Hence $\quad A_0=f(a), A_1=f'(a), A_2=\frac{f''(a)}{2!}, A_3=\frac{f'''(a)}{3!}, \ldots, A_n=\frac{f^n(a)}{n!} \ldots$

Substituting, these values of $A_0, A_1, A_2, \ldots$ in (1), we get

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^n}{n!}f^{(n)}(a)+\ldots \qquad \ldots(2)$$

The power series given by (2) is called the **'Taylor's infinite series** for the expansion of $f(a+h)$ in ascending powers of h.

Cor. 1. Putting $a=x$ in (2), we get

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\ldots+\frac{h^n}{n!}f^{(n)}(x)+\ldots \qquad \ldots(3)$$

Cor. 2. Putting $h=x-a$ in (2), we have

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\ldots+\frac{(x-a)^n}{n!}f^{(n)}(a)+\ldots \qquad \ldots(4)$$

$$=\sum_{k=0}^{\infty}\frac{(x-a)^k}{k!}f^k(a).$$

It is called **Taylor's series of $f(x)$ around $x=a$.**

Remark 1. Putting $a=0$, $h=x$ in equation (2), we get Maclaurin's theorem given by equation (2) of § 1.2-1.

Remark 2. The power series in § 1.2-1 and § 1.2-2 should be convergent.

ILLUSTRATIVE EXAMPLES

Ex. 1. (*a*) *Expand the following by Maclaurin's theorem :*

(*i*) $(a+x)^m$. (*ii*) e^x. (*iii*) $\log(1+x)$.

(*iv*) $\sin x$. (*v*) $\sin^{-1}x$. (*vi*) a^x.

(*vii*) $e^x . \cos x$.

Sol. (*i*) Let

$$f(x)=(a+x)^m, \qquad f(0)=a^m$$

$$f'(x)=m(a+x)^{m-1}, \qquad f'(0)=ma^{m-1}$$

$$f''(x)=m(m-1)(a+x)^{m-2}, \qquad f''(0)=m(m-1)a^{m-2}$$

... ...

$$f^{(n)}(x)=m(m-1)(m-2)\ldots(m-n+1)(a+x)^{m-n}$$

$$f^{(n)}(0)=m(m-1)(m-2)\ldots(m-n+1)a^{m-n} \text{ etc.}$$

Now by Maclaurin's theorem, we have

$$f(x)=f(0)+xf'(0)+(x^2/2!)f''(0)+(x^3/3!)f'''(0)+\ldots$$

Hence $\quad (a+x)^m=a^m+x.ma^{m-1}+(x^2/2!)m(m-1)a^{m-2}+\ldots$
$\quad +(x^n/n!).m(m-1)(m-2)\ldots(m-n+1)a^{m-n}+\ldots$ **Ans.**

(*ii*) Let

$$f(x)=e^x, \qquad f(0)=1$$

$$f'(x)=e^x, \qquad f'(0)=1$$

$$f''(x)=e^x, \qquad f''(0)=1$$

... ...

$$f^{(n)}(x)=e^x, \qquad f^{(n)}(0)=1 \text{ etc.}$$

By Maclaurin's theorem, we get

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots \qquad \textbf{Ans.}$$

(iii) Let

$$f(x) = \log(1+x), \qquad f(0) = 1$$
$$f'(x) = (1+x)^{-1}, \qquad f'(0) = 1$$
$$f''(x) = -(1+x)^{-2}, \qquad f''(0) = -1$$
$$f'''(x) = 2(1+x)^{-3}, \qquad f'''(0) = 2(1)^{-3} = 2!$$
$$f^{(iv)}(x) = -2 \times 3(1+x)^{-4}, \qquad f^{iv}(0) = -(3)!$$
$$\ldots \quad \ldots \quad \ldots$$
$$f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}, \qquad f^{(n)}(0) = (-1)^{n-1}(n-1)! \text{ etc.}$$

∴ By Maclaurin's theorem, we have

$$\log(1+x) = x - (x^2/2) + (x^3/3) - (x^4/4) + \ldots + (-1)^{n-1}(x^n/n) + \ldots .$$ **Ans.**

(iv) Let

$$f(x) = \sin x, \qquad f(0) = \sin 0 = 0$$
$$f'(x) = \cos x, \qquad f'(0) = \cos 0 = 1$$
$$f''(x) = -\sin x, \qquad f''(0) = -\sin 0 = 0$$
$$f'''(x) = -\cos x, \qquad f'''(0) = -\cos 0 = -1$$
$$\ldots \quad \ldots \quad \ldots$$
$$f^{(n)}(x) = \sin\left(x + n \cdot \frac{\pi}{2}\right), \qquad f^{(n)}(0) = \sin \frac{1}{2} n\pi.$$

When n is even, then let $n = 2m$,

∴ $$f^{2m}(0) = 0$$

When n is odd, then let $n = 2m+1$,

∴ $$f^{2m+1}(0) = (-1)^m.$$

∴ By Maclaurin's theorem, we get

$$\sin x = 0 + x \cdot 1 + 0 + \frac{x^3}{3!}(-1) + 0 + \ldots + 0 + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \ldots$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \ldots$$ **Ans.**

(v) Let

$$f(x) = \sin^{-1} x, \; f(0) = 0$$
$$f'(x) = y_1 = \frac{1}{\sqrt{1-x^2}}, \; (y_1)_0 = f'(0) = 1$$

∴ $$y_1^2(1-x^2) = 1.$$

Differentiating again, we have

$$2y_1 y_2 (1-x^2) - 2xy_1^2 = 0 \quad \text{or} \quad (1-x^2)y_2 - xy_1 = 0. \qquad \ldots(1)$$

Differentiating n times w.r.t. 'x', we have

$$(1-x^2)y_{n+2} - (2n+1)y_{n+1} - n^2 y_n = 0.$$

Putting $x = 0$,

$$(y_{n+2})_0 = n^2 (y_n)_0. \qquad \ldots(2)$$

Putting $x = 0$ in (1),

$$(y_2)_0 = 0 = f''(0).$$

Putting $n = 1, 2, 3, \ldots$ in (2), we have

$$(y_3)_0 = 1^2 (y_1)_0 = 1^2 \cdot 1$$
$$(y_4)_0 = 2^2 (y_2)_0 = 0$$
$$(y_5)_0 = 3^2 (y_3)_0 = 3 \cdot 1^2$$
$$\ldots \quad \ldots \quad \ldots$$
$$(y_n)_0 = (n-2)^2 (y_{n-2})_0 = (n-2)^2 (n-4)^2 \ldots 5^2 \cdot 3^2 \cdot 1^2, \text{ if } n \text{ is odd}$$

and $$= 0 \text{ if } n \text{ is even.}$$

By Maclaurin's theorem, we have

$$f(x) = (y)_0 + x(y_1)_0 + (x^2/2!)(y_2)_0 + (x^3/3!)(y_3)_0 + \ldots + (x^n/n!)(y_n)_0 + \ldots$$

∴ $$\sin^{-1} x = 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \cdot 1^2 + \frac{x^4}{4!} \cdot 0 + \frac{x^5}{5!} \cdot 3^2 \cdot 1^2 + \ldots + \frac{x^n}{n!}(n-2)^2(n-4)^2 \ldots 5^2 \cdot 3^2 \cdot 1^2 + \ldots$$
$$= x + 1^2 \frac{x^3}{3!} + 3^2 \cdot 1^2 \frac{x^5}{5!} + \ldots + (n-2)^2(n-4)^2 \ldots 5^2 \cdot 3^2 \cdot 1^2 \cdot \frac{x^n}{n!} + \ldots$$

If n is odd, and the general term is 0 if n is even. **Ans.**

(vi) Let

$$f(x) = a^x, \qquad f(0) = 1$$
$$f'(x) = a^x \log a, \qquad f'(0) = \log a$$
$$f''(x) = a^x (\log a)^2, \qquad f''(0) = (\log a)^2$$
$$f^{(n)}(x) = a^x (\log a)^n, \qquad f^{(n)}(0) = (\log a)^n.$$

By Maclaurin's theorem, we get

$$a^x = 1 + x \log a + \frac{x^2}{2!}(\log a)^2 + \ldots + \frac{x^n}{n!}(\log a)^n + \ldots .$$ **Ans.**

(vii) Let

$$y = e^x \cos x, \qquad (y)_0 = 1$$
$$y_1 = e^x \cos x - e^x \sin x$$

or $$y_1 = y - e^x \sin x, \qquad (y_1)_0 = (y)_0 - 0 = 1$$
$$y_2 = y_1 - y - e^x \sin x, \qquad (y_2)_0 = 1 - 1 = 0$$
$$y_3 = y_2 - y_1 - y - e^x \sin x$$

or $$y_3 = y_2 - 2y, \qquad (y_3)_0 = -2.$$

Again $$y_4 = y_3 - 2y_1, \qquad (y_4)_0 = -2-2 = -2^2$$
$$y_5 = y_4 - 2y_2, \qquad (y_5)_0 = -2^2$$
$$y_6 = y_5 - 2y_3, \qquad (y_6)_0 = 0$$
$$y_7 = y_6 - 2y_4, \qquad (y_7)_0 = 2^3, \text{ etc.}$$

∴ By Maclaurin's theorem, we get

$$e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2 x^4}{4!} - \frac{2^2 x^5}{5!} + \frac{2^3 x^7}{7!} + \ldots .$$

Ex. 1. (*b*) *Show that* $\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - \ldots$ **[R.G.T.U. Jan./Feb. 2007]**

Sol. $$\log(x+h) = \log\left\{h\left(\frac{x}{h}+1\right)\right\} = \log h + \log\left(1 + \frac{x}{h}\right).$$

Put $x/h = y$ and proceed as above Ex. 1 (iii).

Ex. 2. (*a*) *Apply Maclaurin's theorem to prove that* $\log \sec x = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \ldots$

Sol. Let

$$f(x) = \log \sec x, \qquad f(0) = 0$$
$$f'(x) = \frac{\sec x \tan x}{\sec x} = \tan x \qquad f'(0) = 0$$
$$f''(x) = \sec^2 x, \qquad f''(0) = 1$$
$$f'''(x) = 2\sec^2 x \tan x, \qquad f'''(0) = 0$$
$$f^{(iv)}(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x, \qquad f^{(iv)}(0) = 2$$
$$f^{(v)}(x) = 8\sec^2 x \tan^3 x + 16\sec^4 x \tan x, \qquad f^{(v)}(0) = 0$$
$$f^{(vi)}(x) = 16\sec^2 x \tan^4 x + 88\sec^4 x \tan^2 x + 16\sec^6 x, \qquad f^{(vi)}(0) = 16.$$

By Maclaurin's theorem, we have

$$\log \sec x = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0) + \frac{x^6}{6!} f^{vi}(0) + \ldots$$

$$= 0 + x \cdot 0 + \frac{x^2}{2!} \cdot 1 + \frac{x^3}{3!} \cdot 0 + \frac{x^4}{4!} \cdot 2 + \frac{x^5}{5!} \cdot 0 + \frac{x^6}{6!} \cdot 16 + \ldots$$

$$= \frac{1}{2} x^2 + \frac{1}{12} x^4 + \frac{1}{45} x^6 + \ldots$$

Ex. 2. (b) *If* $\log \sec x = \frac{1}{2} x^2 + Ax^4 + Bx^6 + \ldots$, *find A and B.* [R.G.T.U. Dec.

Sol. Proceeding as Ex. 2 (a), we have $A = \frac{1}{12}$ and $B = \frac{1}{45}$. On comparing coefficients of x^4 and x^6.

Ex. 3. *Apply Maclaurin's theorem to prove that* $e^x \sec x = 1 + x + \frac{2x^2}{2!} + \frac{4}{3!} x^3 + \ldots$

Sol. Let $y = e^x \sec x$, $\qquad (y)_0 = 1$

$y_1 = e^x \sec x + e^x . \sec x \tan x = (1 + \tan x)\, y$, $\qquad (y_1)_0 = 1$

$\therefore$ $y_2 = \sec^2 x . y + (1 + \tan x)\, y_1$, $\qquad (y_2)_0 = 2$

$y_3 = \sec^2 x . y_1 + 2 \sec x \sec x \tan x . y + \sec^2 x . y_1 + (1 + \tan x)\, y_2$

$= 2 \sec^2 x . y_1 + 2 \sec^2 x \tan xy + (1 + \tan x)\, y_2 \Rightarrow (y_3)_0 = 4.$

$\therefore$ By Maclaurin's theorem, we have

$$e^x . \sec x = 1 + x + \frac{2x^2}{2!} + \frac{4x^3}{3!} + \ldots .$$

Ex. 4. *Expand* $e^{ax} \cos bx$ *by Maclaurin's theorem. Hence prove that*

$$e^{x \cos \alpha} \cos (x \sin \alpha) = 1 + x \cos \alpha + \frac{x^2}{2!} \cos 2\alpha + \frac{x^3}{3!} \cos 3\alpha + \ldots$$

Sol. Let $y = e^{ax} \cos bx$

$\therefore$ $y_n = r^n e^{ax} . \cos (bx + n\phi)$

where $r = \sqrt{(a^2 + b^2)}$, $\phi = \tan^{-1} \frac{b}{a}$.

$\therefore$ $(y_n)_0 = r^n \cos n\phi$.

Substituting $n = 1, 2, 3, \ldots$ in (1), we get

$$(y_1)_0 = r \cos \phi = (a^2 + b^2)^{1/2} . \frac{a}{\sqrt{a^2 + b^2}} = a$$

$$(y_2)_0 = r^2 \cos 2\phi = (a^2 + b^2) . \frac{1 - \tan^2 \phi}{1 + \tan^2 \phi} = \frac{(a^2 + b^2)(a^2 - b^2)}{a^2 + b^2} = a^2 - b^2,$$

$$(y_3)_0 = r^3 \cos 3\phi = (a^2 + b^2)^{3/2} (4 \cos^3 \phi - 3 \cos \phi) = a(a^2 - 3b^2) \text{ etc.}$$

$$(y_n)_0 = (a^2 + b^2)^{n/2} \cos \{n \tan^{-1} (b/a)\}.$$

Hence by Maclaurin's theorem, we have

$$e^{ax} \cos bx = 1 + ax + (a^2 - b^2) \frac{x^2}{2!} + a(a^2 - 3b^2) . \frac{x^3}{3!} + \ldots$$

For second part, putting $a = \cos \alpha$ and $b = \sin \alpha$, we get

$$e^{x \cos \alpha} \cos (x \sin \alpha) = 1 + x \cos \alpha + \frac{x^2}{2!} \cos 2\alpha + \frac{x^3}{3!} \cos 3\alpha + \ldots .$$

Ex. 5. *Expand* $e^{a \sin^{-1} x}$ *by Maclaurin's theorem and find the general term.*

Hence show that $e^x = 1 + \sin \theta + \frac{1}{2!} \sin^2 \theta + \frac{2}{3!} \sin^3 \theta + \ldots$ **[R.G.T.U. June 2002, June 2008, April 2009]**

Sol. Let $y = e^{a \sin^{-1} x}$, $\qquad (y)_0 = 1$

$\therefore$ $y_1 = e^{a \sin^{-1} x} . \frac{a}{\sqrt{1 - x^2}}$

$\Rightarrow$ $(1 - x^2)\, y_1^2 = a^2 y^2$, $\qquad (y_1)_0 = a.$

Differentiating again, we get

$(1 - x^2)\, 2y_1 y_2 - 2xy_1^2 = a^2 . 2yy_1$

i.e., $(1 - x^2)\, y_2 - xy_1 - a^2 y = 0$, $\qquad (y_2)_0 = a^2.$

Differentiating n times by Leibnitz's theorem, we get

$(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - (n^2 + a^2)\, y_n = 0.$

Putting $x = 0$, $(y_{n+2})_0 = (n^2 + a^2)\, (y_n)_0$. ...(1)

Now substituting $n = 1, 2, 3, 4, 5 \ldots$ in equation (1), we have

$(y_3)_0 = a(1^2 + a^2)$

$(y_4)_0 = a^2(2^2 + a^2)$

$(y_5)_0 = a(1^2 + a^2)(3^2 + a^2).$

$\therefore$ By Maclaurin's theorem, we have

$$e^{a \sin^{-1} x} = 1 + ax + \frac{a^2 x^2}{2!} + \frac{a(1^2 + a^2)\, x^3}{3!} + \ldots . \qquad \ldots(2)$$

Putting $a = 1$, $x = \sin \theta$ in (2), we get

$$e^\theta = 1 + \sin \theta + \sin^2 \theta / 2! + 2 \sin^3 \theta / 3! + \ldots .$$

Ex. 6. *Find the first five terms in the expansion of* $e^{\sin x}$ *by Maclaurin's theorem.* **[R.G.T.U. June 2007]**

Sol. Let $y = e^{\sin x}$, $\qquad (y)_0 = 1.$

Differentiating it successively, we get :

$y_1 = \cos x . e^{\sin x}$,

i.e., $y_1 = y \cos x$, $\qquad (y_1)_0 = 1$

$y_2 = y_1 \cos x - y \sin x$, $\qquad (y_2)_0 = 1$

$y_3 = y_2 \cos x - y_1 \sin x - y_1 \sin x - y \cos x$

$= y_2 \cos x - 2y_1 \sin x - y \cos x$, $\qquad (y_3)_0 = 0$

$y_4 = y_3 \cos x - y_2 \sin x - 2y_2 \sin x - 2y_1 \cos x - y_1 \cos x + y \sin x$

$= y_3 \cos x - 3y_2 \sin x - 3y_1 \cos x + y \sin x$ $\qquad (y_4)_0 = -3$

$y_5 = y_4 \cos x - 4y_3 \sin x - 6y_2 \cos x + 4y_1 \sin x + y \cos x$ $\qquad (y_5)_0 = -8$

$\therefore$ By Maclaurin's theorem,

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} + \ldots .$$

Ex. 7. *Expand* $\tan^{-1} x$ *in ascending powers of x by Maclaurin's theorem.*

Sol. Let $y = \tan^{-1} x$, $\qquad (y)_0 = \tan^{-1} 0 = 0$

$\therefore$ $y_1 = \frac{1}{1 + x^2}$, $\qquad (y_1)_0 = \frac{1}{1 + 0} = 1$

$$(1+x^2)\,y_1=0 \qquad ...(1)$$

or $$(1+x^2)\,y_2+2xy_1=0, \qquad (y_2)_0=0.$$

∴ Differentiating (1) n times by Leibnitz's theorem, we get

$$(1+x^2)\,y_{n+1}+{}^nC_1\,y_n\,(2x)+{}^nC_2\,y_{n-1}\,(2\,.\,1)=0$$

$$(1+x^2)\,y_{n+1}+2nxy_n+n(n-1)\,y_{n-1}=0. \qquad ...(2)$$

or Putting $x=0$, $$(y_{n+1})=-(n-1)\,n\,(y_{n-1})_0.$$

Putting $n=2,3,4,\ldots$ in (2), $$(y_3)_0=-1\,.\,2\,(y_1)_0=-1\,.\,2(1)=-2\,! \qquad [\because\ (y_2)_0=0]$$

$$(y_4)_0=-2\,.\,3\,(y_2)_0=0$$

$$(y_5)_0=-3\,.\,4(y_3)_0=3\,.\,4\,.\,2\,!=4\,!,$$

$$(y_6)_0=0\ (y_7)_0=-5\,.\,6\,(y_5)_0=-5\,.\,6\,.\,4\,!=-6\,!, \text{ etc.}$$

∴ By Maclaurin's theorem, we get

$$\tan^{-1}x=0+x\,.\,1+0+\frac{x^3}{3\,!}(-2\,!)+0+\frac{x^5}{5\,!}(4\,!)+0+\frac{x^7}{7\,!}(-6\,!)+\ldots$$

$$=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots+(-1)^m\frac{x^{2m+1}}{(2m+1)}+\ldots\,.$$

Ex. 8. *Expand by Maclaurins theorem $\frac{e^x}{1+e^x}$ as far as the term x^3.*

Sol. Here $$y=\frac{e^x}{1+e^x}=\frac{1+e^x-1}{1+e^x}=1-\frac{1}{1+e^x}.$$

∴ $$(y)_0=\frac{e^0}{1+e^0}=\frac{1}{2}$$

$$y_1=0+\frac{e^x}{(1+e^x)^2}=\frac{e^x}{1+e^x}\cdot\frac{1}{1+e^x}=y(1-y)=y-y^2,$$

$$(y_1)_0=\frac{1}{4}$$

∴ $$y_2=y_1-2yy_1,$$

∴ $$(y_2)_0=\frac{1}{4}-2\,.\,\frac{1}{2}\cdot\frac{1}{4}=0$$

$$y_3=y_2-2yy_2-2y_1^2,$$

$$(y_3)_0=-1/8 \text{ etc.}$$

∴ By Maclaurin's theorem, we have

$$\frac{e^x}{1+e^x}=\frac{1}{2}+x\,.\,\frac{1}{4}+\frac{x^2}{2\,!}\,.\,0+\frac{x^3}{3\,!}\left(-\frac{1}{8}\right)+\ldots=\frac{1}{2}+\frac{x}{4}-\frac{x^3}{48}+\ldots\,.$$

Ex. 9. *Find the first five terms in the expansion of $\log(1+\sin x)$ by Maclaurin's theorem.* **[R.G.T.U. June 2006]**

Sol. Let $y=\log(1+\sin x)$, then

$$(y)_0=\log(1+0)=0.$$

Now $$e^y=1+\sin x.$$

Differentiating, we get

$$e^y\,.\,y_1=\cos x \qquad ...(1)$$

Putting $x=0$ in (1), we get

$$e^0\,.\,(y_1)_0=1\Rightarrow(y_1)_0=1.$$

Differentiating (1) again, we get

$$e^y\,.\,y_2+e^y\,.\,y_1^2=-\sin x$$

or $$e^y\,.\,y_2+y_1\cos x=-\sin x, \qquad \text{[from (1)]} \quad ...(2)$$

Putting $x=0$ in (2),

$$e^0\,(y_2)_0+1\,.\,1=0\ \Rightarrow\ (y_2)_0=-1.$$

Differentiating (2) again, we get

$$e^y\,y_3+e^y\,.\,y_1\,y_2+y_2\cos x-y_1\sin x=-\cos x$$

or $$e^y\,y_3+y_2\cos x+y_2\cos x-y_1\sin x=-\cos x \qquad \text{[from (1)]}$$

or $$e^y\,y_3+2y_2\cos x-y_1\sin x=-\cos x \qquad ...(3)$$

Putting $x=0$ in (3)

$$1\,.\,(y_3)_0+2(-1)\,.\,1-0=-1\ \Rightarrow\ (y_3)_0=1.$$

Differentiating (3), we have

$$e^y\,y_4+e^y\,y_1y_3+2y_3\cos x-2y_2\sin x-y_2\sin x-y_1\cos x=\sin x$$

or $$e^y y_4+3y_3\cos x-3y_2\sin x-y_1\cos x=\sin x, \qquad \text{[from (1)]} \quad ...(4)$$

Putting $x=0$ in (4), we get

$$e^0\,(y_4)_0+3\,.\,1\,.\,1-0-1\,.\,1=0\Rightarrow(y_4)_0=-2.$$

Differentiating (4), again we get

$$e^y y_5+e^y\,y_1y_4+3y_4\cos x-3y_3\sin x-3y_3\sin x-3y_2\cos x-y_2\cos x+y_1\sin x=\cos x \qquad ...(5)$$

Putting $x=0$ in (5)

$$1\,.\,(y_5)_0+1\,.\,1\,.\,(-2)+3(-2)\,.\,1-0-3(-1)-(-1)+0=1\ \Rightarrow\ (y_5)_0=5.$$

Substituting these values in Maclaurin's theorem, we get

$$\log(1+\sin x)=0+x\,.\,1+\frac{x^2}{2\,!}\,.\,(-1)+\frac{x^3}{3\,!}\,.\,1+\frac{x^4}{4\,!}(-2)+\frac{x^5}{5\,!}\,.\,5+\ldots$$

$$=x-\frac{x^2}{2\,!}+\frac{x^3}{3\,!}-\frac{2x^4}{4\,!}+\frac{5x^5}{5\,!}+\ldots$$

Ex. 10. *Find the first four terms in the expansion of $\log(1+\tan x)$ by Maclaurin's theorem.*

Sol. Let $y=\log(1+\tan x)$, then

$$(y)_0=\log(1+0)=0.$$

Now $$e^y=1+\tan x.$$

By differentiating, we get

$$e^y\,.\,y_1=\sec^2 x. \qquad ...(1)$$

Putting $x=0$ in (1), we get

$$e^0\,.\,(y_1)_0=1\Rightarrow(y_1)_0=1.$$

Differentiating (1), we have

$$e^y y_2+e^y y_1^2=2\sec^2 x\tan x$$

or $$e^y\,(y_2+y_1^2)=2\sec^2 x\tan x. \qquad ...(2)$$

Putting $x=0$ in (2), $$(y_2)_0+1=0\Rightarrow(y_2)_0=-1.$$

Differentiating (2), we get

$$e^y\,(y_3+2y_1\,y_2)+e^y\,.\,y_1\,(y_2+y_1^2)=2\sec^4 x+4\sec^2 x\tan^2 x$$

or $$e^y\,(y_3+3y_1\,y_2+y_1^3)=6\sec^4 x-4\sec^2 x \qquad ...(3)$$

$$[\because\ \tan^2 x=\sec^2 x-1]$$

Putting $x = 0$ in (3), we have

$$(y_3)_0 + 3 \cdot 1 \cdot (-1) + 1 - 6 = 4 \Rightarrow (y_3)_0 = 4.$$

Differentiating (3), we get

$$e^y (y_4 + 3y_2^2 + 3y_1 y_3 + 3y_1^2 \cdot y_2) + e^y \cdot y_1 (y_3 + 3y_1 y_2 + y_1^3) = 24 \sec^4 x \tan x - 8 \sec^2 x \tan x$$

or

$$e^y (y_4 + 4y_1 y_3 + 3y_2^2 + 6y_1^2 y_2 + y_1^4) = (24 \sec^4 x - 8 \sec^2 x) \tan x. \quad ...(4)$$

Putting $x = 0$ in (4),

$$(y_4)_0 + 4 \cdot 1 \cdot 4 + 3(-1)^2 + 6 \cdot 1 (-1) + 1 = 0 \quad \Rightarrow \quad (y_4) = -14.$$

Substituting the values, by Maclaurin's theorem, we get

$$\log (1 + \tan x) = 0 + x \cdot 1 + \frac{x^2}{2!}(-1) + \frac{x^3}{3!} \cdot 4 + \frac{x^4}{4!}(-14) + \ldots = x - \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{7}{12}x^4 + \ldots .$$

Ex. 11. *Expand $\sinh x \cos x$, to fifth powers of x.*

Sol. Here $\quad y = \sinh x \cos x,$

$$(y)_0 = \sinh 0 \cos 0 = 0 \cdot 1 = 0$$

$$y_1 = \cosh x \cos x - \sinh x \sin x. \quad ...(1)$$

Putting $x = 0$ in (1), we have

$$(y_1)_0 = \cosh 0 \cos 0 - \sinh 0 \sin 0$$

or $\quad (y_1)_0 = 1 \cdot 1 - 0 \cdot 0 \Rightarrow (y_1)_0 = 1.$

Differentiating (1), we have

$$y_2 = \sinh x \cos x - \cosh x \sin x - \cosh x \sin x - \sinh x \cos x.$$

or $\quad y_2 = -2 \cosh x \sin x, \quad \therefore \quad (y_2)_0 = 0$

$y_3 = -2 \sinh x \sin x - 2 \cosh x \cos x, \quad \therefore \quad (y_3)_0 = -2$

$y_4 = -2 \cosh x \sin x - 2 \sinh x \cos x + 2 \cosh x \sin x - 2 \sinh x \cos x$

or $\quad y_4 = -4 \sinh x \cos x \quad$ or $\quad y_4 = -4y, \quad \therefore \quad (y_4)_0 = -4(y)_0 = 0$

$y_5 = -4y_1, \quad \therefore \quad (y_5)_0 = -4.$

Substituting the values in Maclaurin's theorem, we have

$$\sinh x \cos x = 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \cdot (-2) + \frac{x^4}{4!} \cdot 0 + \frac{x^5}{5!} \cdot (-4) + \ldots$$

$$= x - (2/3!) x^3 - (4/5!) x^5 + \ldots .$$

Ex. 12. *Expand $e^{x \cos x}$ by Maclaurin's theorem.*

Sol. Let $\quad y = e^{x \cos x}$

$$y_1 = e^{x \cos x} (1 \cdot \cos x - x \sin x) = y (\cos x - x \sin x)$$

$$y_2 = y_1 (\cos x - x \sin x) + y (-\sin x - 1 \cdot \sin x - x \cdot \cos x)$$

$$= y_1 (\cos x - x \sin x) - y (2 \sin x + x \cos x)$$

$$y_3 = y_2 (\cos x - x \sin x) + y_1 (-\sin x - 1 \cdot \sin x - x \cdot \cos x)$$

$$- y_1 (2 \sin x + x \cos x) - y (2 \cos x + 1 \cdot \cos x - x \sin x)$$

$$= y_2 (\cos x - x \sin x) - 2y_1 (2 \sin x + x \cos x) - y(3 \cos x - x \sin x)$$

$$y_4 = y_3 (\cos x - x \sin x) + y_2 (-2 \sin x - x \cos x) - 2y_2 (2 \sin x + x \cos x)$$

$$- 2y_1 (3 \cos x - x \sin x) - y_1 (3 \cos x - x \sin x) - y(- 4 \sin x - x \cos x)$$

$$= y_3 (\cos x - x \sin x) - 3y_2 (2 \sin x + x \cos x)$$

$$- 3y_1 (3 \cos x - x \sin x) + y (4 \sin x + x \cos x)$$

$$y_5 = y_4 (\cos x - x \sin x) - 4y_3 (2 \sin x + x \cos x) - 6y_2 (3 \cos x - x \sin x)$$

$$+ 4y_1 (4 \sin x + x \cos x) + y (5 \cos x - x \sin x).$$

Putting $x = 0$ in above results, we get

$(y)_0 = e^0 = 1;$

$(y_1)_0 = (y)_0 \cdot 1 = 1;$

$(y_2)_0 = (y_1)_0 \cdot 1 = 1;$

$(y_3)_0 = (y_2)_0 \cdot 1 - (y)_0 \cdot 3 = -2;$

$(y_4)_0 = (y_3)_0 - 3(y_1)_0 \cdot 3 = -11;$

$(y_5)_0 = (y_4)_0 \cdot 1 - 0 - 6(y_2)_0 3 + 0 + (y)_0 \cdot 5 = -24$ etc.

Hence substituting these values in Maclaurin's series, we get

$$e^{x \cos x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11x^4}{24} - \frac{x^5}{5} - \ldots . \qquad \textbf{Ans.}$$

Ex. 13. *Prove that* $\quad \log \cosh x = \frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \ldots .$

Sol. Let $\quad y = \log \cosh x. \quad ...(1)$

Therefore $\quad y_1 = \frac{1}{\cosh x} \sinh x = \tanh x \quad ...(2)$

$y_2 = \text{sech}^2 x = 1 - \tanh^2 x = 1 - y_1^2. \quad$ [from (2)] $\quad ...(3)$

$y_3 = -2y_1 y_2 \quad ...(4)$

$y_4 = -2 (y_1 y_3 + y_2^2) \quad ...(5)$

$y_5 = -2(y_1 y_4 + 3y_2 y_3) \quad ...(6)$

$y_6 = -2(y_1 y_5 + 4y_2 y_4 + 3y_3^2). \quad ...(7)$

Putting $x = 0$ in (1), (2), ..., (7), we have

$(y)_0 = \log \cosh 0 = \log 1 = 0;$

$(y_1)_0 = \tanh 0 = 0$

$(y_2)_0 = 1 - (y_1)_0^2 = 1;$

$(y_3)_0 = -2(y_1)_0 (y_2)_0 = 0$

$(y_4)_0 = -2 [(y_1)_0 (y_3)_0 + (y_2)_0^2] = -2(0 + 1) = -2$

$(y_5)_0 = -2 [(y_1)_0 (y_4)_0 + 3(y_2)_0 (y_3)_0] = -2(0) = 0$

$(y_6)_0 = -2 [(y_1)_0 (y_5)_0 + 4(y_2)_0 (y_4)_0 + 3(y_3)_0^2] = 16.$

Hence substituting these values in Maclaurin's Series, we have

$$\log \cosh x = 0 + 0 + \frac{x^2}{2!} \cdot (1) + 0 + \frac{x^4}{4!} \cdot (-2) + 0 + \frac{x^6}{6!} \cdot (16) + \ldots = \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{45} - \ldots .$$

Ex. 14. *Prove that* $(\sin^{-1}x)^2 = \frac{2}{2!}x^2 + \frac{2.2^2}{4!}x^4 + \frac{2.2^2.4^2}{6!}x^6 + \dots$

and hence deduce $\theta^2 = 2.\frac{\sin^2\theta}{2!} + 2^2.\frac{2\sin^4\theta}{4!} + 2^2.4^2.\frac{2\sin^6\theta}{6!} + \dots$

Sol. Let $y = (\sin^{-1}x)^2$, $\quad (y)_0 = (\sin^{-1}0)^2 = 0$

∴ $y_1 = 2(\sin^{-1}x).\frac{1}{\sqrt{1-x^2}}$, $\quad (y_1)_0 = 0$

or $(1-x^2)y_1^2 = 4y$.

Differentiating it, we get

$$(1-x^2)2y_1y_2 - 2xy_1^2 = 4y_1$$

or $(1-x^2)y_2 - xy_1 = 2$, $\quad (y_2)_0 = 2$.

Differentiating n times by Leibnitz's theorem, we get

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0. \quad ...(1)$$

Putting $x = 0$, $\quad (y_{n+2})_0 = n^2(y_n)_0$.

Putting $n = 1, 3, 5, \dots$ in (1), we have $\quad [\because (y_1)_0 = 0]$

$$(y_3)_0 = (y_5)_0 = (y_7)_0 = \dots = 0$$

Putting $n = 2, 4, 6$ in (1), again, we get

$$(y_4)_0 = 2^2(y_2)_0 = 2^2.2$$

$$(y_6)_0 = 4^2.(y_4)_0 = 4^2.2^2.2 \text{ etc.}$$

Hence substituting these values in Maclaurin's theorem, we get

$$(\sin^{-1}x)^2 = 0 + x.0 + \frac{x^2}{2!}.2 + \frac{x^4}{4!}.2^2.2 + \frac{x^5}{5!}.0 + \frac{x^6}{6!}4^2.2^2.2 + \dots$$

$$= \frac{2x^2}{2!} + \frac{2.2^2.x^4}{4!} + \frac{2.2^2.4^2.x^6}{6!} + \dots \quad ...(2)$$

Now putting $x = \sin\theta$ in (2), we get

$$\theta^2 = \frac{2\sin^2\theta}{2!} + 2^2.\frac{2\sin^4\theta}{4!} + 2^2.4^2.\frac{2\sin^6\theta}{6!} + \dots$$

Ex. 15. *Expand* $\sin(a\sin^{-1}x)$ *by Maclaurin's theorem as far as* x^5. *Hence expand* $\sin m\theta$ *in powers of* $\sin\theta$.

Sol. Let $y = \sin(a\sin^{-1}x)$, $\quad \therefore (y)_0 = 0$

∴ $y_1 = \cos(a\sin^{-1}x)\frac{a}{\sqrt{1-x^2}}$ ...(1)

or $\frac{y_1\sqrt{1-x^2}}{a} = \cos(a\sin^{-1}x)$. ...(2)

Squaring (1) and (2) and adding, we get

$$y^2 + \frac{(1-x^2)y_1^2}{a^2} = 1 \text{ or } a^2y^2 + (1-x^2)y_1^2 = a^2.$$

Differentiating again

$$a^2 2yy_1 + (1-x^2)2y_1y_2 + y_1^2(-2x) = 0$$

or $y_2(1-x^2) - xy_1 + a^2y = 0$. ...(3)

Now differentiating each term n times, we have

$$y_{n+2}(1-x^2) + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n(-2) - [y_{n+1}x + {}^nC_1 y_n.1] + a^2y_n = 0$$

or $(1-x^2)y_{n+2} + (2n+1)x.y_{n+1} - (n^2-a^2)y_n = 0$.

Putting $x = 0$, we get

$$(y_{n+2})_0 = (n^2-a^2)(y_n)_0. \quad ...(4)$$

But from (2) and (3), we have

$$(y_1)_0 = a \text{ and } (y_2)_0 = 0.$$

∴ From (4), $\quad (y_2)_0 = 0 = (y_4)_0 = (y_6)_0 = \dots$

∴ **When n is even integer** $(y_n)_0 = 0$.

Again $\quad (y_1)_0 = a$

$$(y_3)_0 = (1^2-a^2)(y_1)_0 = (1^2-a^2)a.$$

Similarly $\quad (y_5)_0 = (3^2-a^2)(1^2-a^2)a.$

∴ **When n is odd integer**

$$(y_n)_0 = [(n-2)^2-a^2][(n-4)^2-a^2]\dots(5^2-a^2)(3^2-a^2)(1^2-a^2)a.$$

Hence by Maclaurin's theorem,

$$\sin(a\sin^{-1}x) = ax + \frac{a(1^2-a^2)}{3!}x^3 + \frac{a(1^2-a^2)(3^2-a^2)x^5}{5!} + \dots$$

Putting $a = m$ and $x = \sin\theta$ on both sides, we have

$$\sin(m\theta) = m\sin\theta + \frac{m(1^2-m^2)}{3!}\sin^3\theta + \frac{m(1^2-m^2)(3^2-m^2)}{5!}\sin^5\theta + \dots$$

Ex. 16. *Expand* $\log\{1 - \log(1-x)\}$ *in powers of x by Maclaurin's theorem as far as the term* x^3. *By substituting* $x/(1+x)$ *for x deduce the expansion of* $\log\{1 + \log(1+x)\}$ *as far as the term in* x^3.

Sol. Let $y = \log\{1 - \log(1-x)\}$

$$e^y = 1 - \log(1-x).$$

Differentiating, we have

$$e^y.y_1 = (1-x)^{-1}. \quad ...(1)$$

Differentiating (1), we get

$$e^y.y_2 + e^y y_1^2 = (1-x)^{-2}. \quad ...(2)$$

Differentiating (2), we get

$$e^y y_3 + e^y.y_1y_2 + e^y.y_1^3 + e^y.2y_1y_2 = 2(1-x)^{-3}. \quad ...(3)$$

Putting $x = 0$ in above results, we get

$$(y)_0 = \log\{1 - \log 1\} = \log(1-0) = 0$$

$$(y_1)_0 = (1-0)^{-1} = 1, (y_2)_0 = 0, (y_3)_0 = 1.$$

Hence by Maclaurin's theorem, we have

$$\log\{1 - \log(1-x)\} = 0 + x.1 + 0 + \frac{x^3}{3!}.1 + \dots = x + \frac{1}{6}x^3 + \dots \quad ...(4)$$

Now substituting $x/(1+x)$ for x on both sides of (4), we have

$$\log\left\{1-\log\left(1-\frac{x}{1+x}\right)\right\}=\frac{x}{1+x}+\frac{1}{6}\left(\frac{x}{1+x}\right)^3+\ldots$$

or

$$\log\{1+\log(1+x)\}=x(1+x)^{-1}+\left(\frac{1}{6}\right)x^3(1+x)^{-3}+\ldots$$

$$=x\{1-x+x^2\ldots\}+\left(\frac{1}{6}\right)x^3\{1+(-3)x+\ldots\}$$

$$=x-x^2+x^3+\ldots+\left(\frac{1}{6}\right)x^3+\ldots=x-x^2+\left(\frac{7}{6}\right)x^3+\ldots.$$ **Ans.**

Ex. 17. *Expand* $\{x+\sqrt{1+x^2}\}^m$ *in ascending powers of* x *and find the general term also.*

Sol. Let

$$y=[x+\sqrt{1+x^2}]^m. \qquad \ldots(1)$$

$\therefore$

$$y_1=m[x+\sqrt{1+x^2}]^{m-1}\left(1+\frac{1}{2}\cdot\frac{2x}{\sqrt{1+x^2}}\right)$$

$$=m[x+\sqrt{1+x^2}]^{m-1}\left(\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\right) \qquad \ldots(2)$$

$\therefore$

$$\sqrt{1+x^2}\,y_1=my$$

or

$$(1+x^2)y_1^2=m^2y^2.$$

Differentiating again,

$$(1+x^2)2y_1y_2+y_1^2.2x=2m^2yy_1 \quad\text{or}\quad y_2(1+x^2)+xy_1=m^2y. \qquad \ldots(3)$$

Now differentiating n times, we have

$$[y_{n+2}(1+x^2)+{}^nC_1\cdot y_{n+1}\cdot 2x+{}^nC_2\cdot y_n\cdot 2]+[\ \ {}_{+1}\cdot x+{}^nC_1\cdot y_n\cdot 1]=m^2y_n$$

or

$$(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+(n^2+m^2)y_n=0.$$

Putting $x=0$, we get

$$(y_{n+2})_0=(m^2-n^2)(y_n)_0. \qquad \ldots(4)$$

But from (1), (2) and (3), we have

$$(y)_0=1,\ (y_1)_0=m.1=m,\ (y_2)_0=m^2.1=m^2.$$

From (4), $\quad (y_4)_0=(m^2-2^2)(y_2)_0=(m^2-2^2).m^2.$

Similarly, $\quad (y_6)_0=(m^2-4^2)(m^2-2^2).m^2.$

$\therefore$ **When n is even then**

$$(y_n)_0=[m^2-(n-2)^2][m^2-(n-4)^2]\ldots(m^2-4^2)(m^2-2^2)m^2. \qquad \ldots(5)$$

Also $\quad (y_1)_0=m.$

Now from (4).

$$(y_3)_0=(m^2-1^2)(y_1)_0=(m^2-1^2)m.$$

Similarly. $\quad (y_5)_0=(m^2-3^2)(m^2-1^2)m.$

$\therefore$ **When n is odd then**

$$(y_n)_0=[m^2-(n-2)^2][m^2-(n-4)^2]\ldots(m^2-3^2)(m^2-1^2)m. \qquad \ldots(6)$$

Now by Maclaurin's theorem, we get

$$y=(y)_0+x(y_1)_0+\frac{x^2}{2!}(y_2)_0+\ldots+\frac{x^n}{n!}(y_n)_0+\ldots$$

$$\therefore\quad [x+\sqrt{1+x^2}]^m=1+mx+\frac{m^2}{2!}x^2+\frac{m(m^2-1^2)}{3!}x^3+\frac{m^2(m^2-2^2)}{4!}x^4+\ldots$$

The general term $=(x^n/n!)(y_n)_0$, where $(y_n)_0$ is given by equations (5) and (6).

Ex. 18. *Expand* $[\log\{x+\sqrt{1+x^2}\}]^2$ *in asscending powers of* x.

Sol. Let

$$y=[\log\{x+\sqrt{1+x^2}\}]^2 \qquad \ldots(1)$$

$\therefore$

$$y_1=2\log[x+\sqrt{1+x^2}]\times\frac{1}{[x+\sqrt{1+x^2}]}\left[1+\frac{1}{2\sqrt{1+x^2}}.2x\right]$$

or

$$y_1=\frac{2\log[x+\sqrt{1+x^2}]}{\sqrt{(1+x^2)}}. \qquad \ldots(2)$$

Squaring both sides, we get

or

$$(1+x^2)y_1^2=4[\log x+\sqrt{1+x^2}]^2=4y \quad\text{or}\quad (1+x^2)y_1^2=4y.$$

Differentiating again, we get

$$(1+x^2)2y_1y_2+y_1^2.2x=4y_1$$

Dividing by $2y_1$, we get

or

$$y_2(1+x^2)+xy_1=2. \qquad \ldots(3)$$

Differentiating each term n times, we get

$$[y_{n+2}(1+x^2)+{}^nC_1.y_{n+1}.2x+{}^nC_2.y_n.2]+[y_{n+1}.x+{}^nC_1.y_n.1]=0$$

or

$$(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+(n^2-n+n)y_n=0$$

or

$$(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+n^2y_n=0.$$

Putting $x=0$, we get

$$(y_{n+2})_0=-n^2(y_n)_0. \qquad \ldots(4)$$

Again when $x=0$, then from (1), (2) and (3), we have

$$(y_1)_0=0$$

$$(y_2)_0(1+0)+0.0=2 \quad\text{or}\quad (y_2)_0=2.$$

From (4), $\quad (y_3)_0=-1^2.(y_1)_0=0 \qquad$ [Taking $n=1$]

$$(y_5)_0=-3^2.(y_3)_0=0,\ \text{etc.}$$

$\therefore$ **When n is odd integer** $(y_n)_0=0.$

Again $\quad (y_2)_0=2,$

$$(y_4)_0=-2^2.(y_2)_0=-2^2.2$$

$$(y_6)_0=-4^2(y_4)_0=-4^2(-2^2).2=(-1)^2\,4^2.2^2.2$$

$$(y_8)_0=-6^2(y_6)_0=-6^2.(-4^2)(-2^2).2=(-1)^3\,6^2.4^2.2^2.2$$

$\therefore$ **When n is even integer, then**

$$(y_n)_0=(-1)^{n/2-1}(n-2)^2(n-4)^2\ldots6^2.4^2.2^2.2.$$

Now by Maclaurin's theorem, we get

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \ldots$$

$$[\log\{x + \sqrt{1+x^2}\}]^2 = 0 + x \cdot 0 + \frac{x^2}{2!} \cdot 2 + \frac{x^3}{3!} \cdot 0 + \frac{x^4}{4!}(-2^2 \cdot 2) + \ldots$$

$$= \frac{2x^2}{2!} - \frac{2 \cdot 2^2 x^4}{4!} + 2 \cdot 2^2 \cdot 4^2 \frac{x^6}{6!} - \ldots$$

Ex. 19. *If* $y = e^{\tan^{-1}x}$ *prove that* $(1+x^2)y_{n+2} + [(2n+1)x - 1]y_{n+1} + n(n+1)y_n = 0$.

Hence or otherwise find out the coefficient of x^5 *if* $e^{\tan^{-1}x}$ *is expanded in powers of* x.

Sol. $y = e^{\tan^{-1}x}$

$\therefore$ $y_1 = e^{\tan^{-1}x} \cdot \frac{1}{1+x^2}$ or $y_1 = \frac{y}{1+x^2}$ or $(1+x^2)y_1 = y$

Differentiating again each term w.r.t. x, we get

$$(1+x^2)y_2 + 2xy_1 = y_1$$

or $$(1+x^2)y_2 + (2x-1)y_1 = 0$$

Now differentiating each term n times by Leibnitz's theorem, we have

$$y_{n+2}(1+x^2) + {}^nC_1 y_{n+1} \cdot 2x + {}^nC_2 y_n \cdot 2 + y_{n+1}(2x-1) + {}^nC_1 y_n \cdot 2 = 0$$

or $$(1+x^2)y_{n+2} + \{(2n+1)x - 1\}y_{n+1} + n(n+1)y_n = 0$$

$$(y)_0 = e^0 = 1 \qquad [\because \tan^{-1}0 = 0]$$

$$(y_1)_0 = \frac{1}{1+0} = 1, \; (y_2)_0 = (y_1)_0 = 1$$

and $$(y_{n+2})_0 = (y_{n+1})_0 - n(n+1)(y_n)_0$$

Putting $n = 1, 2, 3$

$$(y_3)_0 = (y_2)_0 - 1 \cdot 2(y_1)_0 = 1 - 2 \cdot 1 = -1$$

$$(y_4)_0 = (y_3)_0 - 2 \cdot 3(y_2)_0 = -1 - 6(1) = -7$$

$$(y_5)_0 = (y_4)_0 - 3 \cdot 4(y_3)_0 = -7 - 12(-1) = 5$$

Coefficient of x^5 in the expansion of $e^{\tan^{-1}x} = \frac{(y_5)_0}{5!} = \frac{5}{5!} = \frac{1}{24}$

Ex. 20. *Expand* $2x^3 + 7x^2 + x - 1$ *in powers of* $(x - 2)$ *by Taylor's theorem.*

Sol. Let $f(x) = 2x^3 + 7x^2 + x - 1$ ...(1)

We can write $f(x)$ as follows:

$$f(x) = f\{2 + (x-2)\},$$

Now expanding $f\{2 + (x-2)\}$ by Taylor's theorem in powers of $(x - 2)$, we have

$$f(x) = f\{2 + (x-2)\} = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \ldots \quad \ldots(2)$$

Differentiating (1) successively, we have

$$f'(x) = 6x^2 + 14x + 1, \; f''(x) = 12x + 14, \; f'''(x) = 12$$

$$f^{iv}(x) = 0, \text{ hence } f^{(n)}(x) = 0 \text{ when } n \geq 4$$

Putting $x = 2$ in above, we have

$$f(2) = 2 \cdot 2^3 + 7 \cdot 2^2 + 2 - 1 = 45, f'(2) = 53$$

$$f''(2) = 38, f'''(2) = 12, f^{iv}(2) = 0,$$

$$f^{(n)}(2) = 0, \; n \geq 4.$$

Now putting these values in (2), we have

$$f(x) = 45 + (x-2) \cdot 53 + \frac{(x-2)^2}{2!} \cdot 38 + \frac{(x-2)^3}{3!} \cdot 12$$

$$= 45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3.$$

Ex. 21. *Expand* $\sin x$ *in powers of* $(x - \frac{1}{2}\pi)$.

Sol. Let $f(x) = \sin x$, we can write

$$f(x) = f[\tfrac{1}{2}\pi + (x - \tfrac{1}{2}\pi)].$$

By Taylor's theorem, we have

$$f(x) = f[\tfrac{1}{2}\pi + (x - \tfrac{1}{2}\pi)] = f(\tfrac{1}{2}\pi) + (x - \tfrac{1}{2}\pi)f'(\tfrac{1}{2}\pi) + \frac{(x - \frac{1}{2}\pi)^2}{2!}f''(\tfrac{1}{2}\pi) + \ldots \quad \ldots(1)$$

Now $f(x) = \sin x$

$\therefore$ $f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, f^{iv}(x) = \sin x$

$\therefore$ $f(\tfrac{1}{2}\pi) = \sin\tfrac{1}{2}\pi = 1, \; f'(\tfrac{1}{2}\pi) = 0, \; f''(\tfrac{1}{2}\pi) = -1, \; f'''(\tfrac{1}{2}\pi) = 0, \; f^{iv}(\tfrac{1}{2}\pi) = 1.$

Putting these values in (1), we have

$$\sin x = 1 + (x - \tfrac{1}{2}\pi) \cdot 0 + \frac{(x - \frac{1}{2}\pi)^2}{2!} \cdot (-1) + \frac{(x - \frac{1}{2}\pi)^3}{3!} \cdot 0 + \frac{(x - \frac{1}{2}\pi)^4}{4!} \cdot 1 + \ldots$$

$$= 1 - (1/2!)(x - \tfrac{1}{2}\pi)^2 - (1/4!)(x - \tfrac{1}{2}\pi)^4 \ldots$$

Ex. 22. *Expand* $\tan\left(x + \frac{\pi}{4}\right)$ *as far as the term* x^4 *and evaluate* $\tan 46.5°$ *to four significant digits.*

[R.G.T.U. Jan. 2006]

Sol. Let $f(x) = \tan x$ $\qquad \therefore \; f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1.$

Now, $f'(x) = \sec^2 x = 1 + \tan^2 x = 1 + [f(x)]^2$ $\qquad \therefore \; f'\left(\frac{\pi}{4}\right) = 1 + 1 = 2$

$f''(x) = 2f(x)f'(x)$ $\qquad \therefore \; f''\left(\frac{\pi}{4}\right) = 2f\left(\frac{\pi}{4}\right)f'\left(\frac{\pi}{4}\right) = 4$

$$f'''(x) = 2f(x)f''(x) + 2[f'(x)]^2$$

$\therefore$ $$f'''\left(\frac{\pi}{4}\right) = 2f\left(\frac{\pi}{4}\right)f''\left(\frac{\pi}{4}\right) + 2\left[f'\left(\frac{\pi}{4}\right)\right]^2 = 2 \times 1 \times 4 + 2 \times 4 = 16$$

$$f^{iv}(x) = 2f(x)f'''(x) + 6f'(x)f''(x)$$

$\therefore$ $$f^{iv}\left(\frac{\pi}{4}\right) = 2f\left(\frac{\pi}{4}\right)f'''\left(\frac{\pi}{4}\right) + 6f'\left(\frac{\pi}{4}\right)f''\left(\frac{\pi}{4}\right) = 2 \times 1 \times 16 + 6 \times 2 \times 4 = 80.$$

By Taylor's theorem, we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots \quad \ldots(1)$$

Putting $h = x$, $a = \pi/4$ in (1), we get

$$f\left(\frac{\pi}{4}+x\right)=f\left(\frac{\pi}{4}\right)+xf'\left(\frac{\pi}{4}\right)+\frac{x^2}{2!}f''\left(\frac{\pi}{4}\right)+\frac{x^3}{3!}f'''\left(\frac{\pi}{4}\right)+\frac{x^4}{4!}f^{iv}\left(\frac{\pi}{4}\right)$$

$\Rightarrow$ $$\tan\left(x+\frac{\pi}{4}\right)=1+x(2)+\frac{x^2}{2!}(4)+\frac{x^3}{3!}(16)+\frac{x^4}{4!}(80)+\dots$$

$\Rightarrow$ $$\tan\left(x+\frac{\pi}{4}\right)=1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4+\dots \quad \text{...(2)}$$ Ans.

$\Rightarrow$ $$\tan\left(x+\frac{\pi}{4}\right)=1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4+\dots$$

$\Rightarrow$ $$\tan(x+45^\circ)=1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4 \quad \text{...(3)}$$

[Note. $\tan 45^\circ = \tan(\pi/4) = f(\pi/4) = 1$]

Now putting $x = 1{\cdot}5^\circ = \frac{3}{2}\times\frac{\pi}{180}$ radians $= 0{\cdot}02618$ in (3), we get

$$\tan(1{\cdot}5^\circ+45^\circ)=1+2(0{\cdot}02618)+2(0{\cdot}02618)^2+\frac{8}{3}(0{\cdot}02618)^3+\dots$$

$\Rightarrow$ $$\tan 46{\cdot}5^\circ = 1{\cdot}05378$$ Ans.

Ex. 23. *Evaluate $\sin 60^\circ 30'$ correct to five decimal places by using Taylor's theorem. Given $\sin 60^\circ = 0{\cdot}86603$ and $1^\circ = 0{\cdot}01745$ radians.*

Sol. Let $f(x) = \sin x$, $f(x+a) = \sin(x+a)$,

$$f'(x) = \cos x,\ f''(x) = -\sin x,\ f'''(x) = -\cos x,\ f^{iv}(x) = \sin x.$$

$\therefore$ $$f(60^\circ) = \sin 60^\circ = 0{\cdot}86603$$

$$f'(60^\circ) = \cos 60^\circ = 0{\cdot}5,\ f''(60^\circ) = -\sin 60^\circ = -0{\cdot}86603, \text{ etc.}$$

By Taylor's theorem, we have

$$f(x+a) = f(x) + af'(x) + \frac{a^2}{2!}f''(x)+\dots$$

$\Rightarrow$ $$\sin(x+a) = \sin x + a\cos x + \frac{a^2}{2!}(-\sin x) + \frac{a^3}{3!}(-\cos x)+\dots .$$

Putting $x = 60^\circ$, $a = 30' = \frac{1^\circ}{2} = \frac{1}{2}\times 0{\cdot}01745$ radians $= 0{\cdot}008725$ radians.

$$\sin(60^\circ+30') = \sin 60^\circ + 0{\cdot}008725\cos 60^\circ + \frac{(0{\cdot}008725)^2}{2!}(-\sin 60^\circ) + \frac{(0{\cdot}008725)^3}{3!}(-\cos 60^\circ)+\dots$$

$\Rightarrow$ $$\sin 60^\circ 30' = 0{\cdot}86603 + (0{\cdot}008725)(0{\cdot}5) + \frac{(0{\cdot}008725)^2}{2}(-0{\cdot}86603) + \frac{(0{\cdot}008725)^3}{6}(-0{\cdot}5)+\dots$$

$$= 0{\cdot}866030 + 0{\cdot}004363 - 0{\cdot}000033 - 0{\cdot}000000 = 0{\cdot}870360$$

$$= 0{\cdot}87036 \text{ correct upto 5 decimal places.}$$

Ex. 24. *Compute the approximate value of $\sqrt{11}$ to four decimal places by taking the first five terms of an appropriate Taylor's expansion.*

Sol. Let $f(x) = \sqrt{x}$, $\therefore$ $f(x+a) = \sqrt{(x+a)}$.

Now, $$f'(x) = \frac{1}{2}x^{-1/2},\quad f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2} = -\frac{1}{4}x^{-3/2}$$

$$f'''(x) = \frac{3}{8}x^{-5/2},\quad f^{iv}(x) = -\frac{15}{16}x^{-7/2}.$$

By Taylor's series, we have

$$f(x+a) = f(x) + af'(x) + \frac{a^2}{2!}f''(x) + \frac{a^3}{3!}f'''(x)+\dots$$

$\therefore$ $$\sqrt{(x+a)} = \sqrt{x} + a\cdot\frac{1}{2\sqrt{x}} - \frac{a^2}{8}x^{-3/2} + \frac{3a^3}{48}x^{-5/2} - \frac{15a^4}{384}x^{-7/2}.$$

Putting $x = 9$, $a = 2$, we have

$$\sqrt{9+2} = \sqrt{9} + \frac{1}{\sqrt{9}} - \frac{1}{2\times 9\sqrt{9}} + \frac{1}{2\times 81\sqrt{9}} - \frac{5}{8\times 729\sqrt{9}}+\dots$$

$\Rightarrow$ $$\sqrt{11} = 3 + \frac{1}{3} - \frac{1}{54} + \frac{1}{486} - \frac{5}{17496}+\dots$$

$$= 3 + 0{\cdot}33333 - 0{\cdot}01852 + 0{\cdot}00206 - 0{\cdot}00029$$

$$= 3{\cdot}31658 = 3{\cdot}3166 \text{ correct to four decimal places.}$$ Ans.

Ex. 25. *Expand $\sqrt{1+\sin x}$ in powers of x, using Taylor's expansion.*

Sol. We have $$\sqrt{1+\sin x} = \sqrt{1+\cos\left(\frac{\pi}{2}-x\right)} = \sqrt{2}\cos\left(\frac{\pi}{4}-\frac{x}{2}\right).$$

Let $f(h) = \cos h$, so that $f(h+z) = \cos(h+z)$.

$\therefore$ $$f'(h) = -\sin h,\ f''(h) = -\cos h,\ f'''(h) = \sin h.$$

$\Rightarrow$ $$f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}},\ f'\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}},\ f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}},\ f'''\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

By Taylor's series, we have

$$f(h+z) = f(h) + zf'(h) + \frac{z^2}{2!}f''(h) + \frac{z^3}{3!}f'''(h)+\dots .$$

Putting $h = \frac{\pi}{4}$, $z = -\frac{x}{2}$

$$f\left(\frac{\pi}{4}-\frac{x}{2}\right) = \cos\left(\frac{\pi}{4}-\frac{x}{2}\right)$$

$$= f\left(\frac{\pi}{4}\right) + \left(-\frac{x}{2}\right)f'\left(\frac{\pi}{4}\right) + \left(-\frac{x}{2}\right)^2\cdot\frac{1}{2!}f''\left(\frac{\pi}{4}\right) + \left(-\frac{x}{2}\right)^3\cdot\frac{1}{3!}f'''\left(\frac{\pi}{4}\right)+\dots$$

$$= \frac{1}{\sqrt{2}} - \frac{x}{2\sqrt{2}} - \frac{x^2}{8\sqrt{2}} - \frac{x^3}{48\sqrt{2}}+\dots .$$

Hence $$\sqrt{1+\sin x} = \sqrt{2}\cos\left(\frac{\pi}{4}-\frac{x}{2}\right) = \sqrt{2}\left[\frac{1}{\sqrt{2}} - \frac{x}{2\sqrt{2}} - \frac{x^2}{8\sqrt{2}} - \frac{x^3}{48\sqrt{2}}+\dots\right]$$

$$= 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48}+\dots .$$ Ans.

Ex. 26. *Expand $\tan^{-1}x$ in powers of $(x - \frac{1}{4}\pi)$.*

Sol. Let $f(x) = \tan^{-1}x$.

$\therefore$ $$f(x) = f[\tfrac{1}{4}\pi + (x - \tfrac{1}{4}\pi)].$$

$\therefore$ By Taylor's theorem, we have

$$f(x) = f(\tfrac{1}{4}\pi) + (x-\tfrac{1}{4}\pi)f'(\tfrac{1}{4}\pi) + \frac{(x-\frac{1}{4}\pi)^2}{2!}f''(\tfrac{1}{4}\pi)+\dots . \quad \text{...(1)}$$

Now $f(x) = \tan^{-1}x$, $f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$, $f''(x) = -2x/(1+x^2)^2$, etc.

$\therefore$ $f(\frac{1}{4}\pi) = \tan^{-1}\frac{1}{4}\pi$, $f'(\frac{1}{4}\pi) = 1/(1+1/16\pi^2)$, $f''(\frac{1}{4}\pi) = -\pi/[2\{1+(1/16)\pi^2\}^2]$, etc.

Putting these values in (1), the required expansion is given by

$$\tan^{-1}x = \tan^{-1}\left(\frac{\pi}{4}\right) + (x-\tfrac{1}{4}\pi)\frac{1}{(1+\pi^2/16)} - \frac{(x-\frac{1}{4}\pi)^2}{2!}\cdot\frac{\pi}{2(1+\pi^2/16)^2}.$$

Ex. 27. *Expand e^x in powers of $(x-1)$.*

Sol. Let $f(x) = e^x$ then

$$f(x) = f\{1 + (x-1)\}.$$

By Taylor's theorem, we have

$$e^x = f(x) = f\{1 + (x-1)\} = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \ldots \qquad \ldots(1)$$

Now $\qquad f(x) = e^x,\ f'(x) = e^x,\ f''(x) = e^x$ etc.

Therefore $\qquad f(1) = e,\ f'(1) = e,\ f''(1) = e$, etc.

Putting these values in (1), we have

$$e^x = e + (x-1)e + \frac{(x-1)^2}{2!} e + \frac{(x-1)^3}{3!} e + \ldots$$

Ex. 28. *Show that* $\quad \log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} + \ldots$ [R.G.T.U. Jan./Feb. 200

Sol. We are to expand $\log(x+h)$ in ascending powers of x.

So let $f(h) = \log h$ then

$$f(x+h) = \log(x+h).$$

By Taylor's theorem, we have

$$\log(x+h) = f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \ldots \qquad \ldots(1)$$

Here $\qquad f(h) = \log h,$

$\therefore \qquad f'(h) = 1/h,\ f''(h) = -(1/h^2),\ f'''(h) = 2/h^3$, etc.

Substituting these values in (1), we have

$$\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} \ldots$$

Ex. 29. *Expand $\log \sin x$ in powers of $(x-a)$.*

Sol. Let $\qquad f(x) = \log \sin x$

$\therefore \qquad f(x) = f[a + (x-a)] = f(a) + (x-a) f'(a) + (x-a)^2/2!\, f''(a) + \ldots \qquad \ldots(1)$

Now $\qquad f(x) = \log \sin x, \qquad f'(x) = \cot x,$

$f''(x) = -\text{cosec}^2 x, \qquad f'''(x) = 2\,\text{cosec}^2 x \cdot \cot x.$

$\therefore \qquad f(a) = \log \sin a, \qquad f'(a) = \cot a$

$f''(a) = -\text{cosec}^2 a, \qquad f'''(a) = 2\,\text{cosec}^2 a \cot a.$

Substituting these values in (1), we have

$$\log \sin x = \log \sin a + (x-a)\cot a - \frac{(x-a)^2}{2!}\text{cosec}^2 a + \frac{(x-a)^3}{3!} \cdot 2\,\text{cosec}^2 a \cot a + \ldots$$

Ex. 30. *Expand $\log \sin(x+h)$ in powers of h.*

Sol. We are to expand $\log \sin(x+h)$ in powers of h. So let

$$f(x) = \log \sin x, \text{ then } f(x+h) = \log \sin(x+h).$$

$\therefore$ By Taylor's theorem, we get

$$f(x+h) = f(x) + h f'(x) + (h^2/2!) f''(x) + \ldots \qquad \ldots(1)$$

Now $\qquad f(x) = \log \sin x,\ f'(x) = \cot x,\ f''(x) = -\text{cosec}^2 x$, etc.

Substituting these values in (1), we have

$$\log \sin(x+h) = \log \sin x + h \cot x - (h^2/2!)\,\text{cosec}^2 x + (2h^3/3!)\,\text{cosec}^2 x \cot x + \ldots$$

Ex. 31. *Use Taylor's theorem to prove that*

$$\tan^{-1}(x+h) = \tan^{-1} x + h \sin\theta \cdot \frac{\sin\theta}{1} - (h\sin\theta)^2 \frac{\sin 2\theta}{2} + (h\sin\theta)^3 \frac{\sin 3\theta}{3} - \ldots$$

$$+ (-1)^{n-1} (h\sin\theta)^n \frac{\sin n\theta}{n} + \ldots$$

where $\theta = \cot^{-1} x.$ **[R.G.T.U. June 2001]**

Sol. First of all we observe that we are to expand $\tan^{-1}(x+h)$ in powers of h. So let $f(x) = \tan^{-1} x$, then $f(x+h) = \tan^{-1}(x+h)$.

By Taylor's theorem, we have

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \ldots + \frac{h^n}{n!} f^n(x) + \ldots \qquad \ldots(1)$$

Now $\qquad f(x) = \tan^{-1} x$

$$f^{(n)}(x) = D^n(\tan^{-1} x) = (-1)^{n-1}(n-1)!\,\sin^n\theta \sin n\theta$$

where $\theta = \cot^{-1} x$

for $n = 1, \qquad f'(x) = (-1)^0\, 0!\, \sin\theta \sin\theta = \sin\theta \sin\theta$

for $n = 2, \qquad f''(x) = -(1)!\,\sin^2\theta \sin 2\theta$

for $n = 3, \qquad f'''(x) = 2!\,\sin^3\theta \sin 3\theta$ etc.

Substituting these values in (1), we have

$$\tan^{-1}(x+h) = \tan^{-1} x + h \sin\theta \sin\theta - (h^2/2!)\sin^2\theta \sin 2\theta + \frac{h^3}{3!} 2!\,\sin^3\theta \sin 3\theta + \ldots$$

$$+ \frac{h^n}{n!}(-1)^{n-1}(n-1)!\,\sin^n\theta \sin n\theta + \ldots$$

$$= \tan^{-1} x + h\sin\theta \frac{\sin\theta}{1} - (h\sin\theta)^2 \frac{\sin 2\theta}{2} + (h\sin\theta)^3 \frac{\sin 3\theta}{3} - \ldots$$

$$+ (-1)^{n-1}(h\sin\theta)^n \frac{\sin n\theta}{n} + \ldots$$

Ex. 32. *Expand $\log_e x$ in powers of $(x-1)$ and hence evaluate $\log 1.1$ correct to 4 decimal places.* **[R.G.T.U. Dec. 2002]**

Or

Calculate the approximate value of $\log_e 1.1$ correct to 4 decimal places, by using an appropriate Taylor's expansion. **[R.G.T.U. June 2003; Nov/Dec. 2007]**

Sol. Let $\qquad f(x) = \log_e x.$

By Taylor's theorem, we have

$$f(x) = f\{1 + (x-1)\} = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \ldots \qquad \ldots(1)$$

Now $\qquad f'(x) = \frac{1}{x},\ f''(x) = -\frac{1}{x^2},\ f'''(x) = \frac{2}{x^3},\ f^{iv}(x) = \frac{-6}{x^4}.$

$\therefore$ At $x = 1, \qquad f(1) = \log_e 1 = 0,\ f'(1) = 1,\ f''(1) = -1,\ f'''(1) = 2,\ f^{iv}(1) = -6.$

Putting in (1), we get

$$\log_e x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \ldots \qquad \ldots(2)$$

Putting $x = 1{\cdot}1$ *i.e.* $x - 1 = 0{\cdot}1$ in (2), we get

$$\log_e(1{\cdot}1) = (0{\cdot}1) - \frac{(0{\cdot}1)^2}{2} + \frac{(0{\cdot}1)^3}{3} - \frac{(0{\cdot}1)^4}{4} = 0{\cdot}0953. \qquad \textbf{Ans.}$$

PROBLEM SET

1. Expand the following functions in ascending powers of x.
(i) $\tan x$, (ii) $\cos x$, (iii) $\sec x$.

2. (a) Prove that $e^{ax} \sin bx = bx + abx^2 + \frac{3a^2b - b^3}{3!}x^3 + \ldots + \frac{(a^2+b^2)^{n/2}}{n!}x^n \sin(n \tan^{-1}(b/a)) + \ldots$ **[R.G.T.U. Jan./Feb. 200]**

[Hint. Prove as Ex. 4].

3. Prove that $\log(1+e^x) = \log 2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{192}x^4 + \ldots$.

4. Prove that $\log(x+h) = \log_e h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - \ldots$

[Hint : (i) Let
$$f(h) = \log h$$
$$f'(h) = \frac{1}{h}, f''(h) = -\frac{1}{h^2}, f'''(h) = \frac{2}{h^3} \text{ etc.}$$
∴

By Taylor's theorem
$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \ldots$$
or
$$\log(x+h) = \log_e h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - \ldots \Big]$$

5. Prove that $\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \ldots$

[Hint. Let $f(x) = \log x$, now proceed as problem 4 above].

6. Prove that $(x+h)^{-1} = \frac{1}{x} - \frac{h}{x^2} + \frac{h^2}{x^3} - \frac{h^3}{x^4} + \ldots$

[Hint : Let $f(x) = x^{-1}$, ∴ $f(x+h) = (x+h)^{-1}$ etc.]

7. Prove that $\log(1+\sin^2 x) = x^2 - 5/6x^4 + \ldots$

8. Prove that $e^x \sin x = x + x^2 + \frac{2}{3!}x^3 - \frac{2^2}{5!}x^5 - \ldots + \sin\frac{n\pi}{4} \cdot \frac{2^{n/2}}{n!}x^n + \ldots$ **[R.G.T.U. Feb. 20]**

9. Prove that $e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2x^4}{4!} - \frac{2^2x^5}{5!} + \frac{2^3x^7}{7!} + \ldots$

[Hint. See Ex. 1 (vii)].

10. (a) Use Maclaurin's theorem to prove that $\sin(e^x - 1) = x + \frac{1}{2}x^2 - \frac{5}{24}x^4 + \ldots$

(b) If $y = \sin(e^x - 1)$ prove that $x = y - \frac{1}{2}y^2 + \ldots$

[Hint. $y = \sin(e^x - 1) \Rightarrow \sin^{-1} y = e^x - 1 \Rightarrow e^x = 1 + \sin^{-1} y$

Differentiating w.r.t. y, we get
$$e^x x_1 = 1/\sqrt{1-y^2} \qquad \text{(where } x_1 = dx/dy)$$
or
$$(1-y^2)x_1^2 = e^{-2x}$$
Differentiating w.r.t. y, we get
$$(1-y^2)2x_1x_2 - 2yx_1^2 = e^{-2x}(-2x_1)$$
or
$$(1-y^2)x_2 - x_1 y = -e^{-2x}.$$

Putting $y = 0$ in (1), (2) and (3), we get
$$e^{(x)_0} = 1 + 0 \Rightarrow e^{(x)_0} = e^0 \Rightarrow (x)_0 = 0, \quad (x_1)_0 = 1, \quad (x_2)_0 = -1.$$
Hence by Maclaurin's theorem, we get
$$x = (x)_0 + y(x_1)_0 + \frac{y^2}{2!}(x_2)_0 + \ldots = 0 + y \cdot 1 + \frac{y^2}{2!}(-1) + \ldots = y - \frac{1}{2}y^2 + \ldots \Big]$$

11. Expand $\log \sin x$ in powers of $(x-2)$ by Taylor's theorem.
[Hint. See Ex. 27].

12. Expand $2 + x^2 - 3x^5 + 7x^6$ in powers of $(x-1)$. **[R.G.T.U. Dec. 2001]**

13. Find Taylor's series expansions of the function $f(x) = \log \cos x$ about the point $\pi/3$. **[R.G.T.U. Dec. 2003]**

14. Expand $\tan \theta$ in powers of $\left(\theta - \frac{\pi}{4}\right)$. **[R.G.T.U. June 2004]**

15. Find the first four terms in the expansion of $\log_e \sin(x+h)$ in ascending powers of h. Hence calculate the value of $\log_e \sin 31°$ correct to four decimal places. Given $\log_e 2 = 0.69315$.

[Hint. Proceeding as Ex. 32, we have
$$\log \sin(x+h) = \log \sin x + h \cot x - \frac{h^2}{2!}\operatorname{cosec}^2 x + \frac{2h^2}{3!}\operatorname{cosec}^2 x \cot x + \ldots \quad \ldots(1)$$
Now
$$31° = 30° + 1° = 30 \times \frac{\pi}{180} + \frac{\pi}{180} \text{ radians} = \frac{\pi}{6} + \frac{\pi}{180} \text{ radians.}$$
Putting $x = \frac{\pi}{6}$, $h = \frac{\pi}{180}$ in (1), we get
$$\log \sin 31° = \log \sin\left(\frac{\pi}{6} + \frac{\pi}{180}\right)$$
$$= \log \sin\frac{\pi}{6} + \frac{\pi}{180}\cot\frac{\pi}{6} - \frac{1}{2!}\left(\frac{\pi}{180}\right)^2 \operatorname{cosec}^2\frac{\pi}{6} + \frac{2}{3!}\left(\frac{\pi}{180}\right)^3 \operatorname{cosec}^2\frac{\pi}{6}\cot\frac{\pi}{6} + \ldots$$
$$= -\log 2 + \frac{\pi\sqrt{3}}{180} - \frac{1}{2}\left(\frac{\pi}{180}\right)^2 (4) + \frac{1}{3}\left(\frac{\pi}{180}\right)^3 \times 4\sqrt{3} + \ldots$$
$$= -0.69315 + 0.03023 - 0.00061 + 0.00001$$
$$= -0.66352 = -0.6635 \text{ correct to four decimal places.]}$$

16. Calculate the approximate value of $\sqrt{10}$ correct to four decimal places by taking the first four terms of an appropriate Taylor's expansion.
[Hint. Proceed as Ex. 26]

OBJECTIVE TYPE QUESTIONS

Select the Correct Answer :

1. If $f(x)$ be a continuous function and its derivatve upto nth and higher order exists, then in Maclaurin's theorem $f(x) =$:

(a) $f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots$ (b) $f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$

(c) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ (d) $x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$ **[R.G.T.U. 2006 Jan.]**

2. If $f(a+h)$, (where a is independent of h) be a function of h such that it can be expanded in ascending powers of h and this expansion be differentiable any number of times, then $f(a+h) =$:

(a) $f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots$ (b) $f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$

(c) $1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \ldots$ (d) $h + \frac{h^2}{2!}a + \frac{h^3}{3!}a^2 + \ldots$

3. $\sin x =$:

(a) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots$ (b) $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots$

(c) $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots$ (d) $x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots$.

4. $\cos x =$:

(a) $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$ (b) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots$

(c) $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots$ (d) None of these.

5. $\sin^{-1} x =$:

(a) $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots$ (b) $x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots$

(c) $x-1^2.\frac{x^3}{3!}+3^2.1^2\frac{x^5}{5!}-\ldots$ (d) $x+\frac{1^2.x^3}{3!}+3^2.\frac{1^2.x^5}{5!}+\ldots$.

6. Taylor's series expansion of $y=\frac{1}{x}$ about $x=1$ is equal to :

(a) $1-(x-1)+(x-1)^2-(x-1)^3+\ldots$ (b) $1+(x-1)+(x-1)^2+(x-1)^3+\ldots$

(c) $1-(x-1)+\frac{(x-1)^2}{2!}-\frac{(x-1)^3}{3!}+\ldots$ (d) $1+(x-1)+\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+\ldots$.

7. Taylor's series expansion of $y=\sin x$ about $x=\pi/2$ is equal to :

(a) $1-\left(x-\frac{1}{2}\pi\right)^2+\left(x-\frac{1}{2}\pi\right)^4-\ldots$ (b) $1-\frac{1}{2!}\left(x-\frac{1}{2}\pi\right)^2+\frac{1}{4!}\left(x-\frac{1}{2}\pi\right)^4-\ldots$

(c) $\left(x-\frac{\pi}{2}\right)-\frac{1}{3!}\left(x-\frac{1}{2}\pi\right)^3+\frac{1}{5!}\left(x-\frac{1}{2}\pi\right)^5-\ldots$ (d) $\left(x-\frac{\pi}{2}\right)-\left(x-\frac{1}{2}\pi\right)^3+\left(x-\frac{1}{2}\pi\right)^5-\ldots$.

8. The nth term in Maclaurin's series expansion is :

(a) $\frac{f^n(x)}{n!}$ (b) $\frac{f^n(0)}{n!}$

(c) $\frac{f(x)}{n!}$ (d) None of these. **[R.G.T.U. Jan/Feb.]**

Fill in the Blanks by the Expansion of the Given Function :

9. $\sin x =$ .... 10. $\sinh x =$ .... 11. $\cos x =$ .... 12. $\cosh x =$ ....

13. $e^x =$ .... 14. $\sin^{-1} x =$ .... 15. $e^x . \sec x =$ ....

ANSWERS

1. (i) $x+\frac{x^3}{3}+\frac{2}{15}x^5+\ldots$ (ii) $1-\frac{x^2}{2!}+\frac{x^4}{4!}$ (ii) $1+\frac{x^2}{2!}+\frac{5x^4}{4!}+\frac{61x^6}{6!}+\ldots$

11. $\log \sin 2+(x-2)\cot 2-\frac{(x-2)^2}{2!}\text{cosec}^2 2+\frac{(x-2)^3}{3!}.2\,\text{cosec}^2 2 \cot 2+\ldots$,

12. $7+29(x-1)+76(x-1)^2+110(x-1)^3+40(x-1)^4+39(x-1)^5+7(x-1)^6$.

13. $\log\frac{1}{2}-\left(x-\frac{\pi}{3}\right)\sqrt{3}-\frac{4\left(x-\frac{\pi}{3}\right)^2}{2!}-\frac{\left(x-\frac{\pi}{3}\right)^3}{3!}8\sqrt{3}+\ldots$.

14. $1+2\left(\theta-\frac{\pi}{4}\right)+2\left(\theta-\frac{\pi}{4}\right)^2+\ldots$.

15. -0.6635

16. 3.1623.

ANSWERS TO OBJECTIVE TYPE QUESTIONS :

1. (b) 2. (a) 3. (c) 4. (b) 5. (d) 6. (a) 7. (b) 8. (b)

9. $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$ 10. $x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\ldots$ 11. $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$

12. $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ldots$ 13. $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$ 14. $x+\frac{1^2.x^3}{3!}+3^2.1^2\frac{x^5}{5!}+\ldots$

15. $1+x+\frac{2x^2}{2!}+\frac{4x^3}{3!}+\ldots$.

HINTS TO OBJECTIVE TYPE QUESTIONS :

6. Let $f(x)=\frac{1}{x}$, $\therefore\ f'(x)=-x^{-2}, f''(x)=2x^{-3}, f'''(x)=-6x^{-4}$.

Now $f(x)=f\{1+(x-1)\}$

$\Rightarrow$ $\frac{1}{x}=f(1)+(x-1)f'(1)+\frac{(x-1)^2}{2!}f''(1)+\frac{(x-1)^3}{3!}f'''(1)+\ldots$

$=1-(x-1)+(x-1)^2-(x-1)^3+\ldots$.

7. See Ex. 23.

15. See Ex. 3.

Chapter 3

Partial Differentiation, Euler's Theorem and its Application in Approximation and Errors

§ 1.3-1. FUNCTIONS OF TWO AND MORE INDEPENDENT VARIABLES

Consider a rectangle of length x (say) and breadth y (say), then the area A of the rectangle is given by $A = xy$. We note that A depends on x and y. In other words we say that A is a function of two independent variables x and y. Similarly, the volume of a rectangular parallelopiped is a function of three independent variables, namely the length, the breadth and the height. Thus, we may define a function of two independent variables as follows :

Let z be a symbol which has an unique definite value for each pair of values of x and y, then z is said to be a function of two independent variables x and y. It is usually written as $z = f(x, y)$.

A function of two independent variables x and y is also written as $F(x, y)$ or $\psi(x, y)$ or $\phi(x, y)$ etc.

A similar definition can be given for functions of several independent variables.

§ 1.3-2. CONTINUITY OF A FUNCTION OF TWO VARIABLES

Definition. *A function $f(x, y)$ is said to be continuous at a point (x_0, y_0) if for every $\varepsilon > 0$ (where ε is an arbitrarily chosen positive number, however small it may be, but not zero), there exists a corresponding number $\delta > 0$ such that*

$$|x - x_0| < \delta, \ |y - y_0| < \delta$$

$$\Rightarrow \quad |f(x, y) - f(x_0, y_0)| < \varepsilon.$$

§ 1.3-3. PARTIAL DIFFERENTIAL COEFFICIENTS

Let $z = f(x, y)$ be a function of two independent variables x and y. The **partial derivative [or partial differential coefficient]** of z w.r. to 'x' is the ordinary derivative of z w.r. to x when y is regarded as constant. It is written as

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial f}{\partial x} \text{ or } f_x \text{ or } D_x f.$$

Thus $$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

provided this limit exists and is unique.

Similarly the partial derivative of z w.r. to 'y' is the ordinary derivative of z w.r. to y when x is regarded as a constant and is denoted by

$$\frac{\partial z}{\partial y} \text{ or } \frac{\partial f}{\partial y} \text{ or } f_y \text{ or } D_y f.$$

Thus $$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

provided this limit exists and is unique.

Again, if $z = f(x_1, x_2, \ldots, x_n)$ is a function of n independent variables $x_1, x_2, \ldots, x_n$, then the partial derivative of z w.r. to 'x_1' is the ordinary derivative of z w.r. to x_1, when all the other variables namely $x_2, x_3, \ldots, x_n$ are regarded as constants. It is written as $\frac{\partial z}{\partial x_1}$ or $\frac{\partial f}{\partial x_1}$.

Partial Derivatives of Higher Orders :

Let $z = f(x, y)$ be a function of x and y. If $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ exist, then they can be further differentiated partially w.r. to x or/and y, thus giving the second order derivative. We write

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = z_{xx} = f_{xx} = \lim_{h \to 0} \frac{f_x(x+h, y) - f_x(x, y)}{h}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = z_{yy} = f_{yy} = \lim_{k \to 0} \frac{f_y(x, y+k) - f_y(x, y)}{k}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \, \partial x} = f_{yx} = \lim_{k \to 0} \frac{f_x(x, y+k) - f_x(x, y)}{k}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \, \partial y} = f_{xy} = \lim_{h \to 0} \frac{f_y(x+h, y) - f_y(x, y)}{h}.$$

Remark 1. If $f(x, y)$ and its partial derivatives are continuous, then we always have

$$\frac{\partial^2 f}{\partial x \, \partial y} = \frac{\partial^2 f}{\partial y \, \partial x} \quad i.e., \ f_{xy} = f_{yx}.$$

Remark 2. We know that the partial differentiation is the same as the ordinary differentiation when other variables are regarded as constants, hence the following results hold good for partial differentiation.

(i) Partial derivative of a sum.

Let $\phi(x, y)$ and $\psi(x, y)$ be any two functions of x and y. Let $f(x, y) = \phi(x, y) + \psi(x, y)$ *i.e.*, $f = \phi + \psi$, then

$$\frac{\partial f}{\partial x} = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y}.$$

(ii) Partial derivative of the quotient.

Let $f(x, y) = \frac{\phi(x, y)}{\psi(x, y)}$. Then

$$\frac{\partial f}{\partial x} = \frac{\psi \frac{\partial \phi}{\partial x} - \phi \frac{\partial \psi}{\partial x}}{\psi^2}, \quad \frac{\partial f}{\partial y} = \frac{\psi \frac{\partial \phi}{\partial y} - \phi \frac{\partial \psi}{\partial y}}{\psi^2}.$$

(iii) Partial derivative of the product.

Let $f(x, y) = \phi(x, y) \cdot \psi(x, y)$. Then

$$\frac{\partial f}{\partial x} = \phi \frac{\partial \psi}{\partial x} + \psi \frac{\partial \phi}{\partial x}, \quad \frac{\partial f}{\partial y} = \phi \frac{\partial \psi}{\partial y} + \psi \frac{\partial \phi}{\partial y}.$$

(iv) Partial derivative of the function of a function.

Let $z = f(u)$ where $u = u(x, y)$

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$$

Here $\frac{dz}{du}$ is used and not $\frac{\partial z}{\partial u}$, since z is a function of u alone.

ILLUSTRATIVE EXAMPLES

Ex. 1. *If* $F(x, y) = xy + \sin(x + y)$, *find* $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial^2 F}{\partial x^2}, \frac{\partial^2 F}{\partial y^2}, \frac{\partial^2 F}{\partial x \partial y}, \frac{\partial^2 F}{\partial y \partial x}$.

Sol. Here

$$F(x, y) = xy + \sin(x + y)$$

$$\frac{\partial F}{\partial x} = y + \cos(x + y), \quad \frac{\partial F}{\partial y} = x + \cos(x + y);$$

$\therefore$

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\{y + \cos(x + y)\} = -\sin(x + y);$$

$$\frac{\partial^2 F}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial y}\{x + \cos(x + y)\} = -\sin(x + y);$$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial x}\{x + \cos(x + y)\} = 1 - \sin(x + y);$$

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial y}\{y + \cos(x + y)\} = 1 - \sin(x + y).$$

Here it is obvious that

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}.$$

Ex. 2. *If* $f(x, y) = \sin xy^3$, *then find four second order partial derivatives of f.*

Sol. Here $f_x = y^3 \cos xy^3, f_y = 3xy^2 \cos xy^3$; then

$$f_{xx} = -\text{[illegible]};$$

$$f_{xy} = y^3(-3\text{[illegible]} \sin xy^3) + 3y^2 \cos xy^3 = -3xy^5 \sin xy^3 + 3y^2 \cos xy^3;$$

$$f_{yx} = (3xy^2)(-y^3 \sin xy^3) + 3y^2 \cos xy^3 = -3xy^5 \sin xy^3 + 3y^2 \cos xy^3;$$

$$f_{yy} = 3xy^2(-3xy^2 \sin xy^3) + 6xy \cos xy^3 = -9x^2y^4 \sin xy^3 + 6xy \cos xy^3.$$

Clearly $\quad f_{xy} = f_{yx}$

Ex. 3. *Prove that* $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, *where* $f(x, y) = x^3y + e^{xy^2}$

Sol. Here

$$\frac{\partial f}{\partial x} = 3x^2y + y^2e^{xy^2}, \quad \frac{\partial f}{\partial y} = x^3 + 2xye^{xy^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x^3 + 2xye^{xy^2}) = 3x^2 + 2ye^{xy^2} + 2xy^3e^{xy^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(3x^2y + y^2e^{xy^2}) = 3x^2 + 2ye^{xy^2} + 2xy^3e^{xy^2}.$$

Thus

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Ex. 4. *If* $u = \tan^{-1}(y/x) + \sin^{-1}(x/y)$, *prove that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$

Sol.

$$u = \sin^{-1}(x/y) + \tan^{-1}(y/x)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\{1 - (x/y)^2\}^{1/2}} \cdot \frac{1}{y} + \frac{1}{1 + (y/x)^2} \cdot \left(-\frac{y}{x^2}\right) = \frac{1}{(y^2 - x^2)^{1/2}} - \frac{y}{x^2 + y^2}$$

$\therefore$

$$x\frac{\partial u}{\partial x} = \frac{x}{(y^2 - x^2)^{1/2}} - \frac{xy}{x^2 + y^2} \qquad \text{...(1)}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\{1 - (x/y)^2\}^{1/2}} \cdot \left(-\frac{x}{y^2}\right) + \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x}$$

$\therefore$

$$y\frac{\partial u}{\partial y} = \frac{-x}{(y^2 - x^2)^{1/2}} + \frac{xy}{x^2 + y^2} \qquad \text{...(2)}$$

$\therefore$ (1) and (2) $\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$

Ex. 5. *If* $u = ax^2 + 2hxy + by^2$, *find* $\frac{\partial^2 u}{\partial y \partial x}$ *and* $\frac{\partial^2 u}{\partial x \partial y}$.

Sol.

$$u = ax^2 + 2hxy + by^2.$$

$\therefore$

$$\frac{\partial u}{\partial x} = 2ax + 2hy.$$

Hence

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(2ax + 2hy) = 2h.$$

Again

$$\frac{\partial u}{\partial y} = 2hx + 2by.$$

$\therefore$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(2hx + 2by) = 2h.$$

Clearly, we have $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$

Ex. 6. *If* $u = f\left(\frac{y}{x}\right)$ *show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$

Sol. Here

$$u = f\left(\frac{y}{x}\right).$$

$\therefore$

$$\frac{\partial u}{\partial x} = \left\{f'\left(\frac{y}{x}\right)\right\}\left(-\frac{y}{x^2}\right).$$

$\therefore$

$$x\frac{\partial u}{\partial x} = -\frac{y}{x}f'\left(\frac{y}{x}\right). \qquad \text{...(1)}$$

Again

$$\frac{\partial u}{\partial y} = \left\{f'\left(\frac{y}{x}\right)\right\}\left(\frac{1}{x}\right).$$

$\therefore$

$$y\frac{\partial u}{\partial y} = \frac{y}{x}f'\left(\frac{y}{x}\right). \qquad \text{...(2)}$$

(1) and (2) $\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$

Ex. 7. *If* $u = (1 - 2xy + y^2)^{-1/2}$, *prove that* $\frac{\partial}{\partial x}\left\{(1 - x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\} = 0.$

Sol. Here

$$u = (1 - 2xy + y^2)^{-1/2}.$$

$\therefore$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2}(-2y) = yu^3. \qquad \text{...(1)}$$

and

$$\frac{\partial u}{\partial y} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2} \cdot (-2x + 2y) = (x - y)u^3. \qquad \text{...(2)}$$

Now

$$\frac{\partial}{\partial x}\left\{(1 - x^2)\frac{\partial u}{\partial x}\right\} = \frac{\partial}{\partial x}\{(1 - x^2) \cdot yu^3\} = y(-2x)u^3 + y(1 - x^2) \cdot 3u^2\frac{\partial u}{\partial x}$$

$$= -2xyu^3 + 3y(1 - x^2)u^2 yu^3 = -2xyu^3 + 3y^2u^5(1 - x^2). \qquad \text{...(3)}$$

Also $$\frac{\partial}{\partial y}\left\{y^2 \frac{\partial u}{\partial y}\right\} = \frac{\partial}{\partial y}\{y^2 (x-y)\, u^3\} = \frac{\partial}{\partial y}\{(y^2x - y^3)\, u^3\}$$

$$= (2x - 3y^2)\, u^3 + (y^2x - y^3) \,.\, 3u^2 \frac{\partial u}{\partial y}$$

$$= (2xy - 3y^2)\, u^3 + y^2 (x-y)\, 3u^2 (x-y)\, u^3$$

$$= (2xy - 3y^2)\, u^3 + y^2 (x-y)^2 \,.\, 3u^5$$

$$= 2xyu^3 + 3y^2u^5 [(x-y)^2 - u^{-2}] \qquad [\because\ u^{-2} = 1 - 2xy + y^2]$$

$$= 2xyu^3 + 3y^2u^5 [(x-y)^2 - (1 - 2xy + y^2)]$$

$$= 2xyu^3 + 3y^2u^5 [x^2 - 1] = 2xyu^3 - 3y^2u^3 (1 - x^2). \qquad ...(4)$$

(3) and (4)

$\Rightarrow$ $$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2 \frac{\partial u}{\partial y}\right\} = 0.$$

Ex. 8. *If* $u = (1 - 2xy + y^2)^{-1/2}$, *show that* $x\,(\partial u/\partial x) - y\,(\partial u/\partial y) = y^2u^3$.

Sol. Proceeding as in Ex. 7 above, we have from (1) and (2);

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = xyu^3 - y\,(x-y)\, u^3 = y^2u^3.$$

Ex. 9. *If* $z\,(x+y) = x^2 + y^2$, *show that* $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$. [R.G.T.U. Dec. 2000]

Sol. Here $$z = \frac{x^2 + y^2}{x + y}.$$

$\therefore$ $$\frac{\partial z}{\partial x} = \frac{(x+y)\,.\,2x - (x^2 + y^2)\,.\,1}{(x+y)^2} = \frac{x^2 + 2xy - y^2}{(x+y)^2}$$

Similarly $$\frac{\partial z}{\partial y} = \frac{y^2 + 2xy - x^2}{(x+y)^2}.$$

Now $$1 - \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) = 1 - \frac{4xy}{(x+y)^2} = \frac{(x-y)^2}{(x+y)^2}. \qquad ...(1)$$

Again $$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{2\,(x^2 - y^2)}{(x+y)^2} = \frac{2\,(x-y)}{x+y}. \qquad ...(2)$$

(1) and (2) $\Rightarrow$ $$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$

Ex. 10. *If* $f(x, y) = x^3y - y^3x$, *find* $\left(\frac{1}{f_x} + \frac{1}{f_y}\right)_{x=2,\, y=1}$

Sol. Here $$f(x, y) = x^3y - y^3x.$$

$\therefore$ $$f_x = 3x^2y - y^3,\ f_y = x^3 - 3xy^2.$$

Putting $x = 2$, $y = 1$, we have

$$[f_x]_{x=2,\, y=1} = 3\,(2^2)\,.\,1 - (1)^3 = 12 - 1 = 11$$

$$[f_y]_{x=2,\, y=1} = (2)^3 - 3\,(2)\,(1)^2 = 8 - 6 = 2.$$

$\therefore$ $$\left(\frac{1}{f_x} + \frac{1}{f_y}\right)_{x=2,\, y=1} = \frac{1}{11} + \frac{1}{2} = \frac{13}{22}.$$

Ex. 11. *If* $x = u^2 - v^2$, $y = 2uv$, *obtain* $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$.

Sol. Differentiating given relations partially w.r. to 'x', we have

$$1 = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} \qquad ...(1)$$

and $$0 = 2v\frac{\partial u}{\partial x} + 2u\frac{\partial v}{\partial x}. \qquad ...(2)$$

Multiplying (1) by u and (2) by v and then adding, we get

$$u = 2\,(u^2 + v^2)\frac{\partial u}{\partial x} \Rightarrow \frac{\partial u}{\partial x} = \frac{u}{2\,(u^2 + v^2)}.$$

From (2), $$\frac{\partial v}{\partial x} = -\frac{v}{u}\frac{\partial u}{\partial x} = -\frac{v}{2\,(u^2 + v^2)}.$$

Again, differentiating given relations partially w.r. to 'y' and solving, we get

$$\frac{\partial u}{\partial y} = \frac{v}{2\,(u^2 + v^2)} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{u}{2\,(u^2 + v^2)}.$$

Ex. 12. *If* $u = (x^2 + y^2 + z^2)^{-1/2}$; $x^2 + y^2 + z^2 \neq 0$ *then prove that*

(*a*) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u$, [R.G.T.U. Feb. 2005, Jan./Feb. 2007]

(*b*) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 z}{\partial z^2} = 0.$ [R.G.T.U. June 2002, Nov./Dec. 2007]

Sol. (a) Here $$u = \frac{1}{(x^2 + y^2 + z^2)^{1/2}}.$$

Now differentiating both sides partially w.r. to x, we get

$$\frac{\partial u}{\partial x} = -\frac{1}{2\,(x^2 + y^2 + z^2)^{3/2}}\,.\,2x = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}$$

$\therefore$ $$x\frac{\partial u}{\partial x} = -\frac{x^2}{(x^2 + y^2 + z^2)^{3/2}}$$

Similarly, $$y\frac{\partial u}{\partial y} = -\frac{y^2}{(x^2 + y^2 + z^2)^{3/2}}$$

and $$z\frac{\partial u}{\partial z} = -\frac{z^2}{(x^2 + y^2 + z^2)^{3/2}}.$$

Adding (1), (2) and (3), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -\frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}} = -\frac{1}{(x^2 + y^2 + z^2)^{1/2}} = -u.$$

(b) Differentiating $\frac{\partial u}{\partial x}$ partially w. r. to x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left\{-\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right\} = -\left\{\frac{1\,.\,(x^2 + y^2 + z^2)^{3/2} - x\,.\,\frac{3}{2}(x^2 + y^2 + z^2)^{1/2}\,.\,2x}{(x^2 + y^2 + z^2)^3}\right\}$$

$$= -\frac{(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3}\{x^2 + y^2 + z^2 - 3x^2\} = -\frac{1}{(x^2 + y^2 + z^2)^{5/2}}\{y^2 + z^2 - 2x^2\}.$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = -\frac{1}{(x^2 + y^2 + z^2)^{5/2}}\{z^2 + x^2 - 2y^2\}$$

and $$\frac{\partial^2 u}{\partial z^2} = -\frac{1}{(x^2 + y^2 + z^2)^{5/2}}\{x^2 + y^2 - 2z^2\}.$$

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Adding (4), (5) and (6), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-\frac{1}{(x^2+y^2+z^2)^{5/2}}(y^2+z^2-2x^2+z^2+x^2-2y^2+x^2+y^2-2z^2)=0.$$

Ex. 13. *If* $u=x^2\tan^{-1}\frac{y}{x}-y^2\tan^{-1}\frac{x}{y}$, $xy\neq 0$; *then prove that* $\frac{\partial^2 u}{\partial x\,\partial y}=\frac{x^2-y^2}{x^2+y^2}$.

Sol. Here $u=x^2\tan^{-1}\frac{y}{x}-y^2\tan^{-1}\frac{x}{y}$.

Differentiating both sides partially w. r. to y, we have

$$\frac{\partial u}{\partial y}=x^2\cdot\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}-2y\tan^{-1}\frac{x}{y}+y^2\cdot\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{x}{y^2}$$

$$=\frac{x^3}{x^2+y^2}+\frac{xy^2}{x^2+y^2}-2y\tan^{-1}\frac{x}{y}=x-2y\tan^{-1}\frac{x}{y}.$$

Now differentiating $\frac{\partial u}{\partial y}$ partially w.r. to x, we have

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=1-2y\cdot\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{1}{y}=1-\frac{2y^2}{y^2+x^2}=\frac{x^2-y^2}{x^2+y^2}.$$

Ex. 14. *If* $u=\log\frac{x^4+y^4}{x+y}$, *show that* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$. **[R.G.T.U. June 2006]**

Sol. Here $u=\log\frac{x^4+y^4}{x+y}$.

Differentiating both sides partially w.r. to x, we get

$$\frac{\partial u}{\partial x}=\frac{1}{(x^4+y^4)/(x+y)}\cdot\frac{4x^3(x+y)-(x^4+y^4)\cdot 1}{(x+y)^2}=\frac{3x^4+4x^3y-y^4}{(x^4+y^4)(x+y)}.$$

Similarly, $$\frac{\partial u}{\partial y}=\frac{3y^4+4y^3x-x^4}{(x^4+y^4)(x+y)}.$$

$\therefore$ $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{(x^4+y^4)(x+y)}[3x^5+4x^4y-xy^4+3y^5+4y^4x-x^4y]$$

$$=\frac{3[x^5+x^4y+y^4x+y^5]}{(x^4+y^4)(x+y)}=\frac{3(x^4+y^4)(x+y)}{(x^4+y^4)(x+y)}=3.$$

Ex. 15. *If* $u=e^{xyz}$, *show that* $\frac{\partial^3 u}{\partial x\,\partial y\,\partial z}=(1+3xyz+x^2y^2z^2)e^{xyz}$. **[R.G.T.U. June 2001]**

Sol. Here $u=e^{xyz}$.

$\therefore$ $$\frac{\partial u}{\partial z}=e^{xyz}\cdot xy.$$

Again $$\frac{\partial^2 u}{\partial y\,\partial z}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right)=\frac{\partial}{\partial y}[e^{xyz}\cdot xy]=x\frac{\partial}{\partial y}[e^{xyz}\cdot y]$$

$$=x[e^{xyz}+y\cdot e^{xyz}\cdot xz]=x(1+xyz)e^{xyz}=(x+x^2yz)e^{xyz}.$$

Finally, $$\frac{\partial^3 y}{\partial x\,\partial y\,\partial z}=\frac{\partial}{\partial x}\left[\frac{\partial^2 u}{\partial y\partial z}\right]=\frac{\partial}{\partial x}[(x+x^2yz)e^{xyz}]$$

$$=(1+2xyz)e^{xyz}+e^{xyz}\cdot yz(x+x^2yz)$$

$$=[1+2xyz+xyz+x^2y^2z^2]e^{xyz}=[1+3xyz+x^2y^2z^2]e^{xyz}.$$

Ex. 16. *If* $u=\log(x^3+y^3+z^3-3xyz)$, *prove that*

(i) $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3}{x+y+z}$. **[R.G.T.U. June 2007]**

(ii) $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=-\frac{9}{(x+y+z)^2}$. **[R.G.T.U. Jan./Feb. 2008]**

(iii) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-\frac{3}{(x+y+z)^2}$.

Sol. (i) Here $u=\log_e(x^3+y^3+z^3-3xyz)$.

$\therefore$ $$\frac{\partial u}{\partial x}=\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz} \quad ...(1)$$

$$\frac{\partial u}{\partial y}=\frac{3y^2-3xz}{x^3+y^3+z^3-3xyz} \quad ...(2)$$

and $$\frac{\partial u}{\partial z}=\frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}. \quad ...(3)$$

Adding (1), (2) and (3), we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-yz-xz-xy)}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3(x^2+y^2+z^2-yz-xz-xy)}{(x+y+z)(x^2+y^2+z^2-yz-xz-xy)}=\frac{3}{x+y+z}. \quad ...(4)$$

(ii) $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right) \quad \text{[From (4)]}$$

$$=-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}=-\frac{9}{(x+y+z)^2}.$$

(iii) From Algebra, $x^3+y^3+z^3-3xyz=(x+y+z)(x+y\omega+z\omega^2)(x+y\omega^2+z\omega)$

where ω is cube root of unity satisfying $1+\omega+\omega^2=0$ and $\omega^3=1$.

Thus $$u=\log_e(x+y+z)+\log_e(x+y\omega+z\omega^2)+\log_e(x+y\omega^2+z\omega).$$

$\therefore$ $$\frac{\partial u}{\partial x}=\frac{1}{x+y+z}+\frac{1}{x+y\omega+z\omega^2}+\frac{1}{x+y\omega^2+z\omega}$$

$\therefore$ $$\frac{\partial^2 u}{\partial x^2}=-\frac{1}{(x+y+z)^2}-\frac{1}{(x+y\omega+z\omega^2)^2}-\frac{1}{(x+y\omega^2+z\omega)^2} \quad ...(5)$$

Again $$\frac{\partial u}{\partial y}=\frac{1}{x+y+z}+\frac{\omega}{x+y\omega+z\omega^2}+\frac{\omega^2}{x+y\omega^2+z\omega}.$$

$\therefore$ $$\frac{\partial^2 u}{\partial y^2}=-\frac{1}{(x+y+z)^2}-\frac{\omega^2}{(x+y\omega+z\omega^2)^2}-\frac{\omega^4}{(x+y\omega^2+z\omega)^2} \quad ...(6)$$

and $$\frac{\partial u}{\partial z}=\frac{1}{x+y+z}+\frac{\omega^2}{x+y\omega+z\omega^2}+\frac{\omega}{x+y\omega^2+z\omega}$$

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$$\therefore \quad \frac{\partial^2 u}{\partial z^2} = -\frac{1}{(x+y+z)^2} - \frac{\omega^4}{(x+y\omega+z\omega^2)^2} - \frac{\omega^2}{(x+y\omega^2+z\omega)^2}. \qquad ...(7)$$

Adding (5), (6) and (7), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{(x+y+z)^2} - \frac{1+\omega^2+\omega^4}{(x+y\omega+z\omega^2)^2} - \frac{1+\omega^4+\omega^2}{(x+y\omega^2+z\omega)^2}.$$

But

$$1+\omega^2+\omega^4 = 1+\omega^2+\omega = 0 \qquad (\because \omega^4 = \omega^3 \,.\, \omega = \omega)$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{(x+y+z)^2}$$

Ex. 17. *If* $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, *then prove that* $u_x^2 + u_y^2 + u_z^2 = 2\,(xu_x + yu_y + zu_z)$.

Sol. Differentiating both sides of the given relation partially w. r. to x, we get

$$\frac{2x}{a^2+u} - \left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial x} = 0$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\dfrac{2x}{a^2+u}}{\Sigma \dfrac{x^2}{(a^2+u)^2}}. \qquad ...(1)$$

$$\therefore \quad u_x^2 = \left(\frac{\partial u}{\partial x}\right)^2 = \frac{4\dfrac{x^2}{(a^2+u)^2}}{\left\{\Sigma \dfrac{x^2}{(a^2+u)^2}\right\}^2}.$$

$$\therefore \quad u_x^2 + u_y^2 + u_z^2 = \frac{4\Sigma \dfrac{x^2}{(a^2+u)^2}}{\left\{\Sigma \dfrac{x^2}{(a^2+u)^2}\right\}^2} = \frac{4}{\Sigma \dfrac{x^2}{(a^2+u)^2}}. \qquad ...(2)$$

Now multiplying both sides of (1) by $2x$, we get

$$2x\frac{\partial u}{\partial x} = \frac{\dfrac{4x^2}{a^2+u}}{\Sigma \dfrac{x^2}{(a^2+u)^2}}$$

$$\therefore \quad 2\,(xu_x + yu_y + zu_z) = \frac{4\Sigma \dfrac{x^2}{a^2+u}}{\Sigma \dfrac{x^2}{(a^2+u)^2}} = \frac{4}{\Sigma \dfrac{x^2}{(a^2+u)^2}} \qquad ...(3)$$

$$\left\{\because \Sigma \frac{x^2}{a^2+u} = 1\right\}$$

Hence from (2) and (3), we ge[illegible]

$$u_x^2 + u\ [illegible]\ {}_x + yu_y + zu_z).$$

Ex. 18. *If* $\theta = t^n e^{-r^2/4t}$, *what value of n will make* $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$?

Sol. Here

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$$

$$\Rightarrow \quad \frac{1}{r^2}\left[2r\frac{\partial \theta}{\partial r} + r^2\frac{\partial^2 \theta}{\partial r^2}\right] = \frac{\partial \theta}{\partial t} \;\Rightarrow\; \frac{\partial^2 \theta}{\partial r^2} + \frac{2}{r}\frac{\partial \theta}{\partial r} = \frac{\partial \theta}{\partial t}. \qquad ...(1)$$

Now from the given relation $\theta = t^n e^{-r^2/4t}$, we have

$$\frac{\partial \theta}{\partial t} = nt^{n-1}e^{-r^2/4t} + t^n e^{-r^2/4t}\left(\frac{r^2}{4t^2}\right) = e^{-r^2/4t}\, t^{n-2}\left(nt + \frac{r^2}{4}\right)$$

and

$$\frac{\partial \theta}{\partial r} = t^n e^{-r^2/4t}\left(-\frac{r}{2t}\right) = -\frac{r}{2}t^{n-1}e^{-r^2/4t}$$

$$\therefore \quad \frac{\partial^2 \theta}{\partial r^2} = -\frac{t^{n-1}}{2}e^{-r^2/4t} - \frac{r}{2}t^{n-1}e^{-r^2/4t}\left(-\frac{r}{2t}\right) = \frac{t^{n-2}e^{-r^2/4t}}{2}\left[-t + \frac{r^2}{2}\right]$$

$$\therefore \quad \frac{\partial^2 \theta}{\partial r^2} + \frac{2}{r}\frac{\partial \theta}{\partial r} = \frac{t^{n-2}e^{-r^2/4t}}{2}\left[-t + \frac{r^2}{2} + \frac{2}{r}(-tr)\right].$$

Substituting these values in (1), we get

$$\frac{t^{n-2}e^{-r^2/4t}}{2}\left[\frac{r^2}{2} - 3t\right] = t^{n-2}e^{-r^2/4t}\left[nt + \frac{r^2}{4}\right]$$

$$\Rightarrow \quad \frac{r^2}{4} - \frac{3t}{2} = nt + \frac{r^2}{4} \;\Rightarrow\; n = -\frac{3}{2}.$$

Ex. 19. *If* $x^x y^y z^z = c$ *then show that* $\frac{\partial^2 z}{\partial x\,\partial y} = -(x \log ex)^{-1}$. **[R.G.T.U. Dec. 2004]**

Sol. The given relation is

$$x^x y^y z^z = c. \qquad ...(1)$$

Taking logarithm on both sides, we get

$$x \log x + y \log y + z \log z = \log c. \qquad ...(2)$$

Differentiating (2) partially w.r. to x, by noting that z is a function of x and y, we get

$$\left[x\,.\,\frac{1}{x} + \log x\,.\,1\right] + \left[z\,.\,\frac{1}{z} + \log z\,.\,1\right]\frac{\partial z}{\partial x} = 0 \;\Rightarrow\; \frac{\partial z}{\partial x} = -\frac{1+\log x}{1+\log z} \qquad ...(3)$$

Similarly,

$$\frac{\partial z}{\partial y} = -\frac{1+\log y}{1+\log z} \qquad ...(4)$$

Now

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[-\frac{(1+\log y)}{(1+\log z)}\right] = -(1+\log y)\frac{\partial}{\partial x}[(1+\log z)^{-1}]$$

$$= -(1+\log y)\left[-(1+\log z)^{-2}\,.\,\frac{1}{z}\,.\,\frac{\partial z}{\partial x}\right]$$

$$= \frac{(1+\log y)}{z\,(1+\log z)^2}\,.\left\{-\frac{1+\log x}{1+\log z}\right\} \qquad \text{[from (3)]}$$

$\therefore$ At $x = y = z$,

$$\frac{\partial^2 z}{\partial x\,\partial y} = -\frac{(1+\log x)^2}{x(1+\log x)^3} = -\frac{1}{x(1+\log x)} = -\frac{1}{x(\log e + \log x)} \qquad [\because \log e = 1]$$

$$= -\frac{1}{x\log(ex)} = -\{x\log(ex)\}^{-1}.$$

Ex. 20. *If $u = f(r)$, where $r^2 = x^2 + y^2$, show that* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f''(r) + \dfrac{1}{r} f'(r)$. **[R.G.T.U. Dec. 2001, June 2006]**

Sol. It is given that
$$r^2 = x^2 + y^2. \qquad \text{...(1)}$$

Differentiating (1) partially w.r. to x, we get
$$2r\frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r} \qquad \text{...(2)}$$

Similarly,
$$\frac{\partial r}{\partial y} = \frac{y}{r} \qquad \text{...(3)}$$

Now $u = f(r)$ gives
$$\frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x} = f'(r)\frac{x}{r} \qquad \text{[By (2)]}$$

$\therefore$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{f'(r)\,x}{r}\right] = \frac{r\{f'(r).1 + x f''(r).\partial r/\partial x\} - x f'(r).\partial r/\partial x}{r^2}$$

$$= \frac{1}{r^2}\left[r f'(r) + x^2 f''(r) - \frac{x^2}{r} f'(r)\right]$$

Similarly,
$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[r f'(r) + y^2 f''(r) - \frac{y^2}{r} f'(r)\right]$$

Adding, we get
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[2r f'(r) + (x^2+y^2) f''(r) - \frac{(x^2+y^2)}{r} f'(r)\right]$$

$$= \frac{1}{r^2}\left[2r f'(r) + r^2 f''(r) - r f'(r)\right] = \frac{1}{r} f'(r) + f''(r).$$

PROBLEM SET

1. For each of the following functions, verify that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$

(i) $u = \sin^{-1}\dfrac{x}{y}$ (ii) $u = x\cos y + y\cos x$ (iii) $u = \dfrac{xy}{\sqrt{1+x^2+y^2}}$

(iv) $u = \log_e \dfrac{x^2+y^2}{x+y}$ (v) $u = x\log y$ (vi) $u = \log\tan\dfrac{y}{x}$

(vii) $u = \log(y\sin x + x\sin y)$ (viii) $u = \sin xy^2$ (ix) $u = \log\dfrac{xy}{x^2+y^2}$

(x) $u = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$.

2. If $u = \sqrt{x^2+y^2}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u$.

3. If $u = \log\sqrt{x^2+y^2}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$.

4. If $u = e^{x/y}\sin\dfrac{x}{y} + e^{y/x}\cos\dfrac{y}{x}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

5. If $u = \dfrac{xy}{x-y}$, show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$.

6. If $u = e^{\alpha x}\cos\beta x$ and $\beta = \pm\alpha$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

7. If $u = \tan^{-1}\dfrac{xy}{\sqrt{1+x^2+y^2}}$, then prove that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{1}{(1+x^2+y^2)^{3/2}}$

8. If $f(x, y) = x^2\tan^{-1}\left(\dfrac{y}{x}\right) - y^2\tan^{-1}\left(\dfrac{x}{y}\right)$, then prove that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

9. If $u = \sin^{-1}\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, prove that $\dfrac{\partial u}{\partial x} = -\dfrac{y}{x}\cdot\dfrac{\partial u}{\partial y}$

10. If $u = f(y+ax) + \phi(y-ax)$, prove that $\dfrac{\partial^2 u}{\partial x^2} - a^2\dfrac{\partial^2 u}{\partial y^2} = 0$.

11. If $u = x^2+y^2+z^2$, then prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 2u$.

12. If $u = (x^2+y^2+z^2)^{1/2}$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{z}{u}$.

13. If $u = e^{-x}(x\cos y + y\sin x)$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

14. If $u = 3(lx+my+nz)^2 - (x^2+y^2+z^2)$ and $l^2+m^2+n^2 = 1$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$.

15. If $u = \log_e(x^2+y^2+z^2)$, prove that $x\dfrac{\partial^2 u}{\partial y \partial z} = y\dfrac{\partial^2 u}{\partial z \partial x} = z\dfrac{\partial^2 u}{\partial x \partial y}$.

16. If $u = \log_e\sqrt{x^2+y^2+z^2}$, prove that $(x^2+y^2+z^2)\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) = 1$.

17. If $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, prove that $u_x + u_y + u_z = 0$.

18. If $u = x+y+z$, $v = x^2+y^2+z^2$, $w = x^3+y^3+z^3-3xyz$, prove that $\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = 0$.

19. If $\tan u = \dfrac{\cos x}{\sinh y}$ and $\tanh v = \dfrac{\sin x}{\cosh y}$, show that $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

20. If $z = \log(e^x + e^y)$, prove that $rt - s^2 = 0$, where $r = \dfrac{\partial^2 z}{\partial x^2}$, $s = \dfrac{\partial^2 z}{\partial x \partial y}$, $t = \dfrac{\partial^2 z}{\partial y^2}$

§ 1.3-4. TOTAL DIFFERENTIAL COEFFICIENT

Let $u = f(x, y)$ be a function of x and y which possess continuous first order partial derivatives and let $x = \phi(t)$ and $y = \psi(t)$ are functions of t whose first derivatives exist and are continuous with respect to t, where $x = \phi(t)$ and $y = \psi(t)$ are functions of t whose first derivatives exist and are continuous. By substituting the values of x and y in $u = f(x, y)$, u can be regarded as a function of the single variable t, and we can obtain the ordinary differential coefficient du/dt. We call du/dt as the *total differential coefficient* of u, to distinguish it from the partial differential coefficients $\partial u/\partial x$ and $\partial u/\partial y$.

The formulae for du/dt, without first expressing u in terms of t, is given by

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

Generalized Form. If $u=f(x_1,x_2,\ldots x_n)$ and $x_1,x_2,\ldots,x_n$ are all functions of t, then

$$\frac{du}{dt}=\frac{\partial u}{\partial x_1}\cdot\frac{dx_1}{dt}+\frac{\partial u}{\partial x_2}\cdot\frac{dx_2}{dt}+\ldots+\frac{\partial u}{\partial x_n}\cdot\frac{dx_n}{dt}$$

In case, $u=f(x,y)$

where $x=\phi(t_1,t_2)$

and $y=\psi(t_1,t_2)$.

Now if t_2 is regarded as a constant, then x, y as will be functions of t_1 alone.

Therefore from total differential coefficient, we have

$$\frac{\partial u}{\partial t_1}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t_1}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t_1}$$

where the ordinary derivatives have been replaced by the partial derivatives because x, y are functions of the variables t_1 and t_2.

Similarly, regarding t_1 as constant, we have

$$\frac{\partial u}{\partial t_2}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t_2}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t_2}$$

From (4) and (5), we can obtain

$$t_1=F_1(x,y) \quad \text{and} \quad t_2=F_2(x,y).$$

Then

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial x}+\frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial x}$$

and

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial y}+\frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial y}$$

The above results can easily be extended to the case of more than two independent variables.

Differentiation of Implicit Functions :

If $u=f(x,y)$ and $y=\phi(x)$, the total differential coefficient of f w.r. to x is given by

$$\frac{df}{dx}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx} \Rightarrow \frac{df}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx} \quad \ldots(11)$$

(i) First derivatives of an implicit function :

If we have an implicit relation between x and y of the form $f(x,y)=c$, where c is a constant and y is a function of x, (11) becomes

$$0=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx}=-\left(\frac{\partial f}{\partial x}\right)\bigg/\left(\frac{\partial f}{\partial y}\right). \quad \ldots(12)$$

Again, if f is a function of the n variables $x_1,x_2,\ldots,x_n$ and $x_2,x_3,\ldots,x_n$ are all functions of x_1, the total (*i.e.*, the ordinary) differential coefficient of f w.r. to x_1 is given by

$$\frac{df}{dx_1}=\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial x_2}\cdot\frac{dx_2}{dx_1}+\frac{\partial f}{\partial x_3}\cdot\frac{dx_3}{dx_1}+\ldots+\frac{\partial f}{\partial x_n}\cdot\frac{dx_n}{dx_1} \quad \ldots(13)$$

(ii) Second derivative of an implicit function :

Let p, q, r, s and t denote $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x\partial y}$ and $\frac{\partial^2 f}{\partial y^2}$ respectively. Again, let the implicit function be $f(x,y)=c$, then we have

$$\frac{dy}{dx}=-\frac{p}{q}$$

$\therefore$
$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(-\frac{p}{q}\right)=-\left[\frac{q\cdot(dp/dx)-p\cdot(dq/dx)}{q^2}\right]. \quad \ldots(14)$$

But
$$\frac{dp}{dx}=\frac{\partial p}{\partial x}+\frac{\partial p}{\partial y}\cdot\frac{dy}{dx} \quad \text{[using (11)]}$$

$\Rightarrow$
$$\frac{dp}{dx}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{dy}{dx}=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y\partial x}\cdot\frac{dy}{dx} \quad \left[\because \frac{\partial^2 f}{\partial x\,\partial y}=\frac{\partial^2 f}{\partial y\,\partial x}\right]$$

$$=r+s\left(-\frac{p}{q}\right)$$

or
$$\frac{dp}{dx}=\frac{rq-sp}{q} \quad \ldots(15)$$

Similarly,
$$\frac{dq}{dx}=\frac{\partial q}{\partial x}+\frac{\partial q}{\partial y}\cdot\frac{dy}{dx}=s+t\left(-\frac{p}{q}\right)=\frac{sq-tp}{q} \quad \ldots(16)$$

Substituting the values of (15) and (16) in (14), we have

$$\frac{d^2y}{dx^2}=-\frac{q^2r-2pqs+p^2t}{q^3}$$

ILLUSTRATIVE EXAMPLES

Ex. 1. *If* $x^y+y^x=c$, *find* $\frac{dy}{dx}$.

Sol. Let $f(x,y)=x^y+y^x$, then $f(x,y)=c$ which is an implicit function of x and y.

$\therefore$
$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}$$

Now
$$f(x,y)=x^y+y^x.$$

$\therefore$
$$\frac{\partial f}{\partial x}=yx^{y-1}+y^x\log y \quad \text{and} \quad \frac{\partial f}{\partial y}=x^y\log x+xy^{x-1}.$$

$\therefore$
$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\left[\frac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}\right].$$

Ex. 2. *If* $u=x\log xy$, *where* $x^3+y^3+3xy=1$, *find* du/dx.

Sol. Here
$$u=x\log xy. \quad \ldots(1)$$

We know that
$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \quad \ldots(2)$$

Now (1) gives
$$\frac{\partial u}{\partial x}=x\cdot\frac{1}{xy}\cdot y+\log xy=1+\log xy$$

and
$$\frac{\partial u}{\partial y}=x\cdot\frac{1}{xy}\cdot x=\frac{x}{y}$$

Again differentiating the given relation

$$x^3+y^3+3xy=1 \quad \ldots(3)$$

w. r. to x, we get
$$3x^2+3y^2\frac{dy}{dx}+3\left(x\frac{dy}{dx}+y\cdot 1\right)=0 \quad \text{or} \quad \frac{dy}{dx}=-\frac{x^2+y}{y^2+x} \quad \ldots(4)$$

Putting values in (2), we have

$$\frac{du}{dx}=(1+\log xy)+\frac{x}{y}\left[-\frac{(x^2+y)}{(y^2+x)}\right].$$

Ex. 3. *If* $u = x^2 - y^2 + \sin yz$, *where* $y = e^x$ *and* $z = \log_e x$, *find* $\frac{du}{dx}$.

Sol. We have $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dx}$.

Now $u = x^2 - y^2 + \sin yz$ gives

$$\frac{\partial u}{\partial x} = 2x,\ \frac{\partial u}{\partial y} = -2y + z\cos yz,\ \frac{\partial u}{\partial z} = y\cos yz.$$

From $y = e^x$, $\frac{dy}{dx} = e^x$

and from $z = \log_e x$, $\frac{dz}{dx} = (1/x)$.

Substituting these values in (1), we obtain

$$\frac{du}{dx} = 2x + (-2y + z\cos yz) \cdot e^x + y\cos yz \cdot \frac{1}{x}.$$

Ex. 4. *If* $f(x, y) = 0$, $\phi(y, z) = 0$, *show that* $\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y}$.

Sol. If $f(x, y) = 0$, then

$$\frac{dy}{dx} = -\left(\frac{\partial f}{\partial x}\right)\Big/\left(\frac{\partial f}{\partial y}\right) \quad ...(1)$$

and if $\phi(y, z) = 0$, then

$$\frac{dz}{dy} = -\left(\frac{\partial \phi}{\partial y}\right)\Big/\left(\frac{\partial \phi}{\partial z}\right). \quad ...(2)$$

Multiplying (1) and (2), we get

$$\frac{dy}{dx} \cdot \frac{dz}{dy} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right)\Big/\left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial \phi}{\partial z}\right)$$

or

$$\left(\frac{\partial f}{\partial y}\right) \cdot \left(\frac{\partial \phi}{\partial z}\right) \cdot \frac{dz}{dx} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right).$$

Ex. 5. *If x, y, z are connected by the equations* $\phi(x, y, z) = 0$ *and* $\psi(x, y, z) = 0$, *find* $\frac{dy}{dx}$.

Sol. Since $\phi(x, y, z) = 0$.

$\therefore$ $\frac{d\phi}{dx} = 0$.

Hence $\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = 0.$ ...(1)

Similarly, $\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial \psi}{\partial z} \cdot \frac{dz}{dx} = 0.$ ...(2)

Multiplying (1) by $\frac{\partial \psi}{\partial z}$ and (2) by $\frac{\partial \phi}{\partial z}$ and then subtracting, we get

$$\left\{\frac{\partial \phi}{\partial x} \cdot \frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial x} \cdot \frac{\partial \phi}{\partial z}\right\} + \left\{\frac{\partial \phi}{\partial y} \cdot \frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial x} \cdot \frac{\partial \phi}{\partial z}\right\}\frac{dy}{dx} = 0.$$

$$\therefore \quad \frac{dy}{dx} = \frac{\left\{\frac{\partial \phi}{\partial z} \cdot \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial z} \cdot \frac{\partial \phi}{\partial x}\right\}}{\left\{\frac{\partial \psi}{\partial z} \cdot \frac{\partial \phi}{\partial y} - \frac{\partial \phi}{\partial z} \cdot \frac{\partial \psi}{\partial x}\right\}}$$

Ex. 6. *If* $a\ x^2 + 2hxy + by^2 = 1$, *find* $\frac{d^2y}{dx^2}$.

Sol. Let $f(x, y) = ax^2 + 2hxy + by^2 = 1$, then

$$p = \frac{\partial f}{\partial x} = 2(ax + hy),\ q = \frac{\partial f}{\partial y} = 2(hx + by),\ r = \frac{\partial^2 f}{\partial x^2} = 2a,\ s = \frac{\partial^2 f}{\partial x \partial y} = 2h,\ t = \frac{\partial^2 f}{\partial y^2} = 2b.$$

Hence $\frac{d^2y}{dx^2} = -\frac{q^2 \cdot r - 2qps + p^2 t}{q^3}$ gives

$$\frac{d^2y}{dx^2} = -\frac{2^2(hx + by)^2 \cdot 2a - 2 \cdot 2(hx + by) \cdot 2(ax + hy) \cdot 2h + 2^2(ax + hy)^2 \cdot 2b}{2^3(hx + by)^3}$$

$$= -\frac{(hx + by)^2 \cdot a - 2(hx + by)(ax + hy)h + (ax + hy)^2 b}{(hx + by)^3}.$$

Ex. 7. (*a*) *If* $u = f(y - z, z - x, x - y)$, *prove that* $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Sol. Let $X = y - z,\ Y = z - x$ and $Z = x - y$. ...(1)

Then $u = f(X, Y, Z)$, where each one of X, Y, Z is a function of x, y, z.

From (1), we have

$$\frac{\partial X}{\partial x} = 0,\ \frac{\partial X}{\partial y} = 1,\ \frac{\partial X}{\partial z} = -1;\ \frac{\partial Y}{\partial x} = -1,\ \frac{\partial Y}{\partial y} = 0,\ \frac{\partial Y}{\partial z} = 1;$$

and

$$\frac{\partial Z}{\partial x} = 1,\ \frac{\partial Z}{\partial y} = -1,\ \frac{\partial Z}{\partial z} = 0.$$

Now

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial x}$$

$$= \frac{\partial u}{\partial X} \cdot 0 + \frac{\partial u}{\partial Y} \cdot (-1) + \frac{\partial u}{\partial Z} \cdot 1 = -\frac{\partial u}{\partial Y} + \frac{\partial u}{\partial Z} \quad ...(2)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial y}$$

$$= \frac{\partial u}{\partial X} \cdot 1 + \frac{\partial u}{\partial Y} \cdot 0 + \frac{\partial u}{\partial Z} \cdot (-1) = \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z} \quad ...(3)$$

and

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial z}$$

$$= \frac{\partial u}{\partial X} \cdot (-1) + \frac{\partial u}{\partial Y} \cdot 1 + \frac{\partial u}{\partial Z} \cdot 0 = -\frac{\partial u}{\partial X} + \frac{\partial u}{\partial Y} \quad ...(4)$$

Adding (2), (3) and (4), we have

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Ex. 7. (*b*) *If* $u = u\left(\frac{y - x}{xy}, \frac{z - x}{xz}\right)$, *show that* $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$. **[R.G.T.U. 2000 Summer]**

Sol. Let $X = \frac{y - x}{xy} = \frac{1}{x} - \frac{1}{y}$ and $Y = \frac{z - x}{xz} = \frac{1}{x} - \frac{1}{z}$.

Then $u = u(X, Y)$, where each one of X, Y is a function of x, y, z.

Now, $\frac{\partial X}{\partial x} = -\frac{1}{x^2},\ \frac{\partial X}{\partial y} = \frac{1}{y^2},\ \frac{\partial X}{\partial z} = 0$ and $\frac{\partial Y}{\partial x} = -\frac{1}{x^2},\ \frac{\partial Y}{\partial y} = 0,\ \frac{\partial Y}{\partial z} = \frac{1}{z^2}$

We have, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial x} = \frac{\partial u}{\partial X}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial Y}\left(-\frac{1}{x^2}\right)$

or $$x^2 \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial X} - \frac{\partial u}{\partial Y} \quad ...(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y} = \frac{\partial u}{\partial X}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial y}(0)$$

or $$y^2 \frac{\partial u}{\partial y} = \frac{\partial u}{\partial X} \quad ...(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial z} = \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y}\left(\frac{1}{z^2}\right)$$

or $$z^2 \frac{\partial u}{\partial z} = \frac{\partial u}{\partial Y} \quad ...(3)$$

Adding (1), (2) and (3), we get

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0.$$

Ex. 7. (c) *If* $f(cx - az, cy - bz) = 0$, *show that* $ap + bq = c$, *where* $p = \frac{\partial z}{\partial x}$ *and* $q = \frac{\partial z}{\partial y}$.

Sol. Let $X = cx - az,\ Y = cy - bz.$

Then $f(X, Y) = 0$, where each one of X, Y is a function of x, y, z. Taking z as a function of x and y. We have

$$\frac{\partial X}{\partial x} = c - a\frac{\partial z}{\partial x},\quad \frac{\partial X}{\partial y} = -a\frac{\partial z}{\partial y},\quad \frac{\partial Y}{\partial x} = -b\frac{\partial z}{\partial x},\quad \frac{\partial Y}{\partial y} = c - b\frac{\partial z}{\partial y}.$$

We know that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial x}$$

or $$0 = \frac{\partial f}{\partial X}\left(c - a\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial Y}\left(-b\frac{\partial z}{\partial x}\right) \quad [\because f(cx - az, cy - bz) = 0]$$

$$0 = c\frac{\partial f}{\partial X} - \frac{\partial z}{\partial x}\left(a\frac{\partial f}{\partial X} + b\frac{\partial f}{\partial Y}\right) \quad \text{or} \quad \frac{\partial z}{\partial x} = \frac{c\frac{\partial f}{\partial X}}{a\frac{\partial f}{\partial X} + b\frac{\partial f}{\partial Y}} \quad ...(1)$$

Again $$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial y} \quad \text{or} \quad 0 = \frac{\partial f}{\partial X}\left(-a\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial Y}\left(c - b\frac{\partial z}{\partial y}\right)$$

or $$0 = c\frac{\partial f}{\partial Y} - \frac{\partial z}{\partial y}\left(a\frac{\partial f}{\partial X} + b\frac{\partial f}{\partial Y}\right) \quad \text{or} \quad \frac{\partial z}{\partial y} = \frac{c\frac{\partial f}{\partial Y}}{a\frac{\partial f}{\partial X} + b\frac{\partial f}{\partial Y}} \quad ...(2)$$

Multiplying (1) by a and (2) by b, then adding, we get

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = \frac{c\left(a\frac{\partial f}{\partial X} + b\frac{\partial f}{\partial Y}\right)}{a\frac{\partial f}{\partial X} + b\frac{\partial f}{\partial Y}} \quad \text{or} \quad ap + bq = c.$$

Ex. 8. *If* $u = xyz$, *evaluate* d^2u.

Sol. Since u is a function of x, y and z, we have

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}y + \frac{\partial u}{\partial z}dz$$

$$\therefore \quad d \equiv dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}.$$

Now $$d^2u = d\,(du) = \left(dx\frac{\partial}{\partial u} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right)\cdot\left(dx\frac{\partial u}{\partial x} + dy\frac{\partial u}{\partial y} + dz\frac{\partial u}{\partial z}\right)$$

$$= (dx)^2\frac{\partial^2 u}{\partial x^2} + 2\,dx\,dy\frac{\partial^2 u}{\partial x\,\partial y} + 2dx\,dz\frac{\partial^2 u}{\partial z\,\partial x} + (dy)^2\frac{\partial^2 u}{\partial y^2} + 2\,dy\,dz\frac{\partial^2 u}{\partial y\,\partial z} + (dz)^2\frac{\partial^2 u}{\partial z^2}. \quad ...(1)$$

Again $$u = xyz \Rightarrow \frac{\partial u}{\partial x} = yz,\ \frac{\partial u}{\partial y} = zx,\ \frac{\partial u}{\partial z} = xy$$

$$\frac{\partial^2 u}{\partial x^2} = 0,\ \frac{\partial^2 u}{\partial y^2} = 0,\ \frac{\partial^2 u}{\partial z^2} = 0$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = z,\ \frac{\partial^2 u}{\partial y\,\partial z} = x,\ \frac{\partial^2 u}{\partial z\,\partial x} = y.$$

Putting these values in (1), we get

$$d^2u = 2\,(z\,dx\,dy + y\,dz\,dx + x\,dy\,dz).$$

§ 1.3-5. HOMOGENEOUS FUNCTION

A function is called *homogeneous function* if the degree of each term is same.

Let $$f(x, y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_ny^n$$

be a function of x and y.

Obviously each term of the function $f(x, y)$ is of degree n. Thus $f(x, y)$ is a *homogeneous function of degree n in x and y.*

Now $f(x, y)$ can be written as

$$f(x, y) = x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \ldots + a_n\left(\frac{y}{x}\right)^n\right] = x^n F\left(\frac{y}{x}\right).$$

Again, $$f(x, y) = y^n\left[a_0\left(\frac{x}{y}\right)^n + a_1\left(\frac{x}{y}\right)^{n-1} + a_2\left(\frac{x}{y}\right)^{n-2} + \ldots + a_n\right] = y^n\phi\left(\frac{x}{y}\right).$$

Thus, every homogeneous function of degree n in x and y can be written as either $x^n F\left(\frac{y}{x}\right)$ or $y^n\phi\left(\frac{x}{y}\right)$.

Definition. A function $f(x, y)$ is said to be a *homogeneous function of degree n in x and y, if*

$$f(tx, ty) = t^n f(x, y).$$

Examples : (i) $\tan^{-1}(x/y)$ is a homogeneous function of degree 0.

(ii) $x^3 + x^2y + y^3$ is a homogeneous function of degree 3.

(iii) If f is a homogeneous function of x and y of degree n, then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are homogeneous functions of x and y of degree $(n-1)$ each.

§ 1.3-6. EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

Theorem. *If* $f(x, y)$ *is a homogeneous function of x and y of degree n, then* $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.$

Proof. Since f is a homogeneous function of x and y of degree n, we can write

$$f(x, y) = x^n F\left(\frac{y}{x}\right)$$

Then
$$\frac{\partial f}{\partial x} = x^n F'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + nx^{n-1} F\left(\frac{y}{x}\right)$$

$\Rightarrow$
$$x\frac{\partial f}{\partial x} = -x^{n-1} y F'\left(\frac{y}{x}\right) + nx^n F\left(\frac{y}{x}\right).$$

Again,
$$\frac{\partial f}{\partial y} = x^n F'\left(\frac{y}{x}\right)\cdot\frac{1}{x} \Rightarrow y\frac{\partial f}{\partial y} = x^{n-1} y F'\left(\frac{y}{x}\right).$$

Adding (2) and (3), we have
$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n F\left(\frac{y}{x}\right) \Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.$$

Generalized Form. Euler's Theorem can be extended to a homogeneous function of any numbe[r of vari]ables. Thus, if $f(x_1, x_2, \ldots, x_n)$ be a homogeneous function of n variables, say, $x_1, x_2, \ldots, x_n$ of degre[e n, then]

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \ldots + x_n\frac{\partial f}{\partial x_n} = nf.$$

§ 1.3-7. RELATIONS BETWEEN SECOND ORDER DERIVATIVES OF HOMOGENEO[US] FUNCTIONS

Theorem. *If u is a homogeneous function of degree n, then*

(i) $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial x \partial y} = (n-1)\dfrac{\partial u}{\partial x}$. (ii) $x\dfrac{\partial^2 u}{\partial x \partial y} + y\dfrac{\partial^2 u}{\partial y^2} = (n-1)\dfrac{\partial u}{\partial y}$.

(iii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = n(n-1)u$.

Proof. Since u is a homogeneous function of x and y of degree n, therefore by Euler's Theorem, w[e have]

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Differenting both sides of (1) partially w. r. to x, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = n\frac{\partial u}{\partial x}$$

$\Rightarrow$
$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial x}$$

Again differentiating both sides of (1) partially w. r. to y, we get

$$x\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = n\frac{\partial u}{\partial y}$$

$\Rightarrow$
$$x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$$

Multiplying (2) by x and (3) by y and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = n(n-1)u$$

[using (1)]

§ 1.3-8. AN IMPORTANT DEDUCTION FROM EULER'S THEOREM

Theorem. *If $u = \phi(F_n)$ where F_n is a homogeneous function of degree n, and suppose that this relation implies $F_n = f(u)$, then*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}.$$

Proof. Since $F_n = f(u)$ is a homogeneous function of degree n, therefore by Euler's Theorem, we have

$$x\frac{\partial}{\partial x}f(u) + y\frac{\partial}{\partial y}f(u) = nf(u)$$

$\Rightarrow$
$$xf'(u)\frac{\partial u}{\partial x} + yf'(u)\frac{\partial u}{\partial y} = nf(u) \Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}.$$

ILLUSTRATIVE EXAMPLES

Ex. 1. *If $u = \tan^{-1}\left(\dfrac{x^3 + y^3}{x - y}\right)$ then show that*

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$. **[R.G.T.U. Dec. 2002]**

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + y^2\dfrac{\partial^2 u}{\partial y^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} = \sin 4u - \sin 2u$ **[R.G.T.U. Jan./Feb. 2006]**

$= 2\cos 3u \sin u$. **[R.G.T.U. Dec. 2003, Jan./Feb. 2008, April 2009]**

Sol. (i) Here
$$u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right).$$

$\therefore$
$$\tan u = \frac{x^3 + y^3}{x - y} = f(x, y), \text{ say.}$$

Again,
$$f(x, y) = \frac{x^3 + y^3}{x - y} = \frac{x^3[1 + (y/x)^3]}{x[1 - (y/x)]} = x^2 F(y/x).$$

Thus f is a homogeneous function of x and y of degree 2. Hence by Euler's theorem on homogeneous function, we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f.$$

$\therefore$
$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u \qquad [\because f = \tan u]$$

$\Rightarrow$
$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u \Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

$\therefore$
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u. \qquad \ldots(1)$$

(ii) Differentiating (1) partially w.r. to x, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = 2\cos 2u\frac{\partial u}{\partial x}$$

$\Rightarrow$
$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (2\cos 2u - 1)\frac{\partial u}{\partial x}. \qquad \ldots(2)$$

Again differentiating (1) partially w.r. to y, we get

$$x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 2\cos 2u\frac{\partial u}{\partial y}$$

$$\Rightarrow \quad x\frac{\partial^2 u}{\partial x\,\partial y}+y\frac{\partial^2 u}{\partial y^2}=(2\cos 2u-1)\frac{\partial u}{\partial y}.$$

Multiplying (2) by x and (3) by y and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial y^2}+y^2\frac{\partial^2 u}{\partial y^2}=(2\cos 2u-1)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=(2\cos 2u-1)\sin 2u$$

$$=2\cos 2u\sin 2u-\sin 2u=\sin 4u-\sin 2u=2\cos 3u\sin u.$$ [using (

Ex. 2. *If* $u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, *then show that* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u.$ **[R.G.T.U. June 2004, Nov./Dec. 20**

Sol. Here $\quad u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right).$

$$\therefore \quad \sin u=\frac{x^2+y^2}{x+y}=f(x,y),\ \text{say}.$$

Again, $\quad f(x,y)=\frac{x^2\,[1+(y/x)^2]}{x\,[1+(y/x)]}=x^1\,F\left(\frac{y}{x}\right).$

Thus f is a homogeneous function of x and y of degree 1. Hence by Euler's theorem, we have

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=1\,.\,f$$

$$\Rightarrow \quad x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)=1\,.\,\sin u$$

$$\Rightarrow \quad x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\sin u.$$

$$\therefore \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u.$$

Ex. 3. *If* $V=\log_e \sin\left\{\frac{\pi\,(2x^2+y^2+xz)^{1/2}}{2\,(x^2+xy+2yz+z^2)^{1/3}}\right\}$ *find the value of* $x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}$ *at* $x=0,y=1,z=$

Sol. Let $\quad u=\frac{\pi\,(2x^2+y^2+xz)^{1/2}}{2\,(x^2+xy+2yz+z^2)^{1/3}}$

Then $\quad V=\log_e \sin u$

$$\therefore \quad \frac{\partial V}{\partial x}=\cot u\,\frac{\partial u}{\partial x}$$

$$\frac{\partial V}{\partial y}=\cot u\,\frac{\partial u}{\partial y}$$

and $\quad \frac{\partial V}{\partial z}=\cot u\,\frac{\partial u}{\partial z}.$

Multiplying (2), (3) and (4) by x, y and z respectively and adding, we get

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}=\cot u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right].$$

Now $\quad u=\frac{\pi x\,[2+(y/x)^2+(z/x)]^{1/2}}{2x^{2/3}\,[1+(y/x)+2\,(y/x)\,(z/x)+(z/x)^2]^{1/3}}=x^{1/3}\,F\left(\frac{y}{x},\frac{z}{x}\right).$

Thus u is a homogeneous function of x, y and z of degree $\frac{1}{3}$. Hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{1}{3}u.$$

Putting this value in (5), we get

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}=\frac{1}{3}u\cot u.$$

When $x=0, y=1, z=2$, we have $u=\pi/4$.

$$\therefore \quad x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}=\frac{1}{3}\cdot\frac{\pi}{4}\cot\frac{\pi}{4}=\frac{\pi}{12}.$$

Ex. 4. *If* $u=\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$, *then verify Euler's theorem for the function* u.

Sol. Here $\quad u=\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}=\frac{x^{1/4}\,[1+(y/x)^{1/4}]}{x^{1/5}\,[1+(y/x)^{1/5}]}=x^{1/20}\,F(y/x).$

Thus u is a homogeneous function of x and y of degree 1/20.
Hence to verify Euler's theorem for u, we have to prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{20}u.$$

Now $$u=\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}. \quad ...(1)$$

$$\therefore \quad \frac{\partial u}{\partial x}=\frac{(x^{1/5}+y^{1/5})\,(\frac{1}{4}x^{-3/4})-(x^{1/4}+y^{1/4})\,(\frac{1}{5}x^{-4/5})}{(x^{1/5}+y^{1/5})^2}$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x}=\frac{(x^{1/5}+y^{1/5})\,(\frac{1}{4}x^{1/4})-(x^{1/4}+y^{1/4})\,(\frac{1}{5}x^{1/5})}{(x^{1/5}+y^{1/5})^2} \quad ...(2)$$

Again from (1), we have

$$\frac{\partial u}{\partial y}=\frac{(x^{1/5}+y^{1/5})\,(\frac{1}{4}y^{-3/4})-(x^{1/4}+y^{1/4})\,(\frac{1}{5}y^{-4/5})}{(x^{1/5}+y^{1/5})^2}$$

$$\Rightarrow \quad y\frac{\partial u}{\partial y}=\frac{(x^{1/5}+y^{1/5})\,(\frac{1}{4}y^{1/4})-(x^{1/4}+y^{1/4})\,(\frac{1}{5}y^{1/5})}{(x^{1/5}+y^{1/5})^2} \quad ...(3)$$

Adding (2) and (3), we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{(x^{1/5}+y^{1/5})\,(\frac{1}{4}x^{1/4}+\frac{1}{4}y^{1/4})-(x^{1/4}+y^{1/4})\,(\frac{1}{5}x^{1/5}+\frac{1}{5}y^{1/5})}{(x^{1/5}+y^{1/5})^2}$$

$$=\frac{(\frac{1}{4}-\frac{1}{5})\,(x^{1/5}+y^{1/5})\,(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}=\frac{1}{20}\cdot\frac{(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})}=\frac{1}{20}u.$$

Hence Euler's theorem is verified for the given function u.

Ex. 5. *If* $u=\sin^{-1}\left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right)^{1/2}$ *prove that* $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{144}(13+\tan^2 u).$

Sol. Here $\quad u=\sin^{-1}\left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right)^{1/2} \Rightarrow \sin u=\left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right)^{1/2}$

$$\Rightarrow \quad \sin u=x^{-1/12}\left[\frac{1+(y/x)^{1/3}}{1+(y/x)^{1/2}}\right]^{1/2} \Rightarrow \sin u=x^{-1/12}\,F(y/x)=f(x,y),\ \text{say}.$$

Ex. 7. *If* $u = \dfrac{xy}{x+y}$, *show that* $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 0.$

Sol. Here
$$u = \frac{x^2 (y/x)}{x(1 + y/x)} = x f\left(\frac{y}{x}\right).$$

Now $u = x f(y/x)$ is a homogeneous function of x and y of degree 1, so from relation (iii) of § 1.3-7, we have

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 1(1-1)u = 0.$$

Ex. 8. *If* $u = (x+y)/(x^3+y^3)$, *applying Euler's theorem, prove that the degree of u is* -2.

Sol. Here
$$u = (x+y)/(x^3+y^3) \quad ...(1)$$

$$\frac{\partial u}{\partial x} = \frac{(x^3+y^3) \cdot 1 - (x+y) \cdot 3x^2}{(x^3+y^3)^2}$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} = \frac{(x^3+y^3) \cdot x - (x+y) \cdot 3x^3}{(x^3+y^3)^2} \quad ...(2)$$

$$\frac{\partial u}{\partial y} = \frac{(x^3+y^3) \cdot 1 - (x+y) \cdot 3y^2}{(x^3+y^3)^2}$$

$$\Rightarrow \quad y\frac{\partial u}{\partial y} = \frac{(x^3+y^3) \cdot y - (x+y) \cdot 3y^3}{(x^3+y^3)^2}. \quad ...(3)$$

Adding (2) and (3), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{(x^3+y^3)(x+y) - (x+y) \cdot 3(x^3+y^3)}{(x^3+y^3)^2} = \frac{-2(x+y)(x^3+y^3)}{(x^3+y^3)^2} = \frac{-2(x+y)}{(x^3+y^3)} = -2u.$$

Hence the degree of the given homogeneous function is -2.

PROBLEM SET

1. Find dy/dx, if
   (i) $e^x + e^y = 2xy$. (ii) $ax^2 + 2hxy + by^2 = 1$. (iii) $(\tan x)^y + (y)^{\cot x} = a$.
2. If $\sqrt{(1-x^2)} + \sqrt{(1-y^2)} = a(x-y)$, prove that $\dfrac{dy}{dx} = \dfrac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}$.

   [Hint. $[\sqrt{(1-x^2)} + \sqrt{(1-y^2)}]/(x-y) = a$].
3. If $u = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$ show that $\dfrac{du}{dt} = \dfrac{3}{\sqrt{1-t^2}}$.
4. If $u = x \log(xy)$ where $x^3 + y^3 + 3xy = 1$, prove that $\dfrac{du}{dx} = 1 + \log(xy) - \dfrac{x(x^2+y)}{y(x+y^2)}$.
5. If $u = \sqrt{x^2+y^2}$ and $x^3 + y^3 + 3axy = 5a^2$, find the value of $\dfrac{du}{dx}$ when $x = a$, $y = a$.
6. If $u = x^4 y^5$ where $x = t^2$ and $y = t^3$, find du/dt and prove it by direct substitution.
7. If $u = \sin^{-1} \dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.
8. If $u = \dfrac{x^3 \cdot y^3}{x^3 + y^3}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3u$.

Chapter 4

Maxima and Minima of Function of Two Variables

§ 1.4-0. MAXIMA AND MINIMA OF FUNCTION OF ONE VARIABLE

Dear students, you have read, maxima and minima of one variable, in your intermediate classes. Here I am giving this topic in brief for your ready reference.

Working Method. Let $f(x)$ [$=y$] be the given function of one independent variable x. Then the procedure is

(i) Find $\frac{dy}{dx}$ *i.e.*, $f'(x)$.

(ii) Put $f'(x)=0$ and solve it for x. Let $x=a, b, c, \ldots$.
Let $x=a$ be one of these values.

(iii) Find $f''(x)$ and hence find $f''(a)$.

(iv) If $f''(a)=-$ve, then $x=a$ is a point of maxima

(v) If $f''(x)=+$ve then $x=a$ is a point of minima.

(vi) If $f''(a)=0$ then find $f'''(a)$. If $f'''(a)\neq 0$, then there is neither maximum nor minimum at $x=a$. if $f'''(a)=0$ then find $f^{iv}(a)$. If $f^{iv}(a)$ is negative, we have a maximum at $x=a$.

If $f^{iv}(a)$ is positive, we have a minimum at $x=a$.

If $f^{iv}(a)$ is also equal to zero, then find $f^{v}(a), f^{vi}(a)$ and so on.

Maxima and Minima in a Closed Interval [a, b] :

Let $f(x)$ be the given function which is defined in the closed interval $[a, b]$.

Find maximum and minimum values of the function at points in $[a, b]$.

Let maximum value of $f(x)=M$

and let minimum value of $f(x)=m$.

Also find $f(a)$ and $f(b)$. Then

(i) Maximum value of $f(x)$ in $[a, b]$ = Largest of $f(a), M, f(b)$

(ii) Minimum value of $f(x)$ in $[a, b]$ = Smallest of $f(a), m, f(b)$.

ILLUSTRATIVE EXAMPLES

Ex. 1. *Show that the function* $x^5-5x^4+5x^3-1$ *is maximum at* $x=1$, *minimum at* $x=3$ *and neither when* $x=0$.

Sol. Let $y=x^5-5x^4+5x^3-1$.

$\therefore$ $dy/dx=5x^4-20x^3+15x^2$

$d^2y/dx^2=20x^3-60x^2+30x.$ ...(1)

For maxima and minima of y, put $dy/dx=0$.

$\therefore$ $5x^4-20x^3+15x^2=0 \Rightarrow 5x^2(x-1)(x-3)=0 \Rightarrow x=0, 1, 3.$

At $x=1$. $d^2y/dx^2=-10=-\text{ve}$.

$\therefore$ y is maximum at $x=1$.

At $x=3$. $d^2y/dx^2=20\times 27-60\times 9+30\times 3=90$.

$\therefore$ y is minimum at $x=1$.

At $x=0$. $d^2y/dx^2=0$.

Differentiating (1), $d^3y/dx^3=60x^2-120x+30=30\neq 0$ at $x=0$.

Since $d^2y/dx^2=0$ and $d^3y/dx^3\neq 0$ at $x=0$.

Hence y has neither maxima nor minima at $x=0$.

Ex. 2. *Find the largest and smallest values of* x^3-18x^2+96x *in the interval* $[0, 9]$. **[R.G.T.U. Jan./Feb. 20**

Sol. Let

$$f(x)=x^3-18x^2+96x.$$

$\therefore$

$$f'(x)=3x^2-36x+96$$

$$f''(x)=6x-36.$$

For maxima and minima of $f(x)$, put $f'(x)=0$ *i.e.,*

$$3x^2-36x+96=0 \Rightarrow 3(x-4)(x-8)=0 \Rightarrow x=4, 8.$$

At $x=4$. $f''(4)=6\times 4-36=-12=-\text{ve}$.

$\therefore$ $f(x)$ is maximum at $x=4$.

Maximum value of $f(x)=f(4)=(4)^3-18(4)^2+96(4)=160$.

At $x=8$. $f''(8)=6\times 8-36=12=+\text{ve}$.

$\therefore$ $f(x)$ is minimum at $x=8$.

Minimum value of $f(x)=f(8)=(8)^3-18(8)^2+96(8)=128$.

Clearly both the points $x=4$, $x=8$ belong to $[0, 9]$.

Also $f(0)=0$

and $f(9)=9^3-18\times 9^2+96\times 9=135$.

From results (1) to (4), that in the interval $[0, 9]$;

largest value of $f(x)=160$,

and smallest value of $f(x)=0$.

Ex. 3. *Show that* $\sin x(1+\cos x)$ *is a maximum when* $x=\pi/3$. **[R.G.T.U. June 20**

Sol. Let

$$y=\sin x(1+\cos x)=\sin x+\frac{1}{2}\sin 2x.$$

$\therefore$

$$dy/dx=\cos x+\cos 2x,\ d^2y/dx^2=-\sin x-2\sin 2x.$$

For maxima and minima, put $dy/dx=0$

i.e., $\cos x+\cos 2x=0 \Rightarrow \cos x=-\cos 2x$

$\Rightarrow$ $\cos x=\cos(\pi-2x) \Rightarrow x=\pi-2x \Rightarrow x=\pi/3$.

At $x=\pi/3$,

$$\frac{d^2y}{dx^2}=-\sin\frac{\pi}{3}-2\sin\frac{2\pi}{3}=-\frac{\sqrt{3}}{2}-\frac{2.\sqrt{3}}{2}$$

$$=-\frac{3}{2}\sqrt{3}=\text{a negative quantity.}$$

Hence the given function is a maximum at $x=\pi/3$.

Ex. 4. *Show that the function* $\sin 3x-3\sin x$ *is minimum when* $x=\frac{\pi}{2}$ *and maximum when* $x=\frac{3\pi}{2}$. **[R.G.T.U. April 20**

Sol. Let

$$f(x)=\sin 3x-3\sin x$$

$\therefore$

$$f'(x)=3\cos 3x-3\cos x$$

$$f''(x)=-9\sin 3x+3\sin x.$$

For max. and min. $f'(x)=0 \Rightarrow 3\cos 3x-3\cos x=0$

$\Rightarrow$ $\cos 3x-\cos x=0 \Rightarrow 4\cos^3 x-4\cos x$

$\Rightarrow$ $\cos x(\cos^2 x-1)=0 \Rightarrow \cos x=0 \Rightarrow x=\frac{\pi}{2}, \frac{3\pi}{2}$.

[Neglecting $\cos^2 x=1$ *i.e.,* $x=0, \pi$ since at these points $f(x)=$

At $x=\frac{\pi}{2}$, $f''\left(\frac{\pi}{2}\right)=12=+\text{ve}$.

$\therefore$ $f(x)$ is minimum at $x=\pi/2$.

At $x=\frac{3\pi}{2}$,

$$f''\left(\frac{3\pi}{2}\right)=-9\sin\frac{9\pi}{2}+3\sin\frac{3\pi}{2}$$

$$=-9\sin\frac{\pi}{2}-3\sin\frac{\pi}{2}$$

$$=-12=-\text{ve}.$$

$\therefore$ $f(x)$ is maximum at $x=3\pi/2$.

Ex. 5. *Find the volume of the largest possible right circular cylinder that can be inscribed in a sphere of radius a.* **[R.G.T.U. June 2007**

Sol. Let O be the centre of the sphere of radius a.

Let h be the height and r the radius of cylinder which is inscribed in the sphere. (See figure)

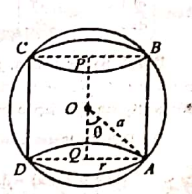

Fig. 1.1

Let $\angle AOQ=\theta$.

Then $h=PQ=2OQ=2OA\cos\theta=2a\cos\theta$

$r=a\sin\theta$.

The volume V of the cylinder is given by

$$V=\pi r^2 h=\pi(a\sin\theta)^2(2a\cos\theta)=2\pi a^3\sin^2\theta\cos\theta.$$

$\therefore$

$$\frac{dV}{d\theta}=2\pi a^3[\sin^2\theta(-\sin\theta)+\cos\theta\,.\,2\sin\theta\cos\theta]$$

$$=2\pi a^3[2\sin\theta\cos^2\theta-\sin^3\theta]$$

and

$$\frac{d^2V}{d\theta^2}=2\pi a^3(2\cos^3\theta-7\sin^2\theta\cos\theta).$$

Now $\frac{dV}{d\theta}=0 \Rightarrow \sin\theta(2\cos^2\theta-\sin^2\theta)=0 \Rightarrow \sin\theta=0$ or $\tan\theta=\pm\sqrt{2}$.

Thus when $\tan\theta=\sqrt{2}$ then $\frac{d^2V}{d\theta^2}=$ negative *i.e.,* V is maximum,

[the other values being inadmissible]

When $\tan\theta=\sqrt{2}$ then $\sin\theta=\sqrt{(2/3)}$, $\cos=1/\sqrt{3}$.

$\therefore$ Maximum volume $V=2\pi a^3\left(\frac{2}{3}\right)\left(\frac{1}{\sqrt{3}}\right)=\frac{4\pi a^3}{3\sqrt{3}}$.

Note. Height of largest cylinder $=h=\frac{2a}{\sqrt{3}}$

MAXIMA AND MINIMA OF FUNCTIONS OF TWO INDEPENDENT VARIABLES

§ 1.4-1. MAXIMA AND MINIMA OF FUNCTIONS OF TWO INDEPENDENT VARIABLES

Let $f(x, y)$ be a function of two independent variables x and y and let $f(x, y)$ be continuous for all values of x and y in the small neighbourhood of (a, b).

Maximum. The function $f(x, y)$ of two independent variables x and y is said to have a *maximum* at the point (a, b), if

$$f(a+h, b+k) < f(a, b)$$

for all sufficient small independent values of h and k, positive or negative.

Minimum. The function $f(x, y)$ of two independent variables x and y is said to have *minimum* at the point (a, b), if

$$f(a+h, b+k) > f(a, b)$$

for all sufficient small independent values of h and k, positive or negative.

§ 1.4-2. NECESSARY CONDITIONS FOR THE EXISTENCE OF MAXIMA OR MINIMA OF $f(x, y)$ AT (a, b)

Theorem 1. *The necessary conditions for the existence of a maxima or a minima of $f(x, y)$ at $x = a$ and $y = b$ are*

$$\left(\frac{\partial f}{\partial x}\right)_{(a, b)} = 0 \text{ and } \left(\frac{\partial f}{\partial y}\right)_{(a, b)} = 0$$

where $\left(\frac{\partial f}{\partial x}\right)_{(a, b)}$ and $\left(\frac{\partial f}{\partial y}\right)_{(a, b)}$ respectively denote the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $x = a$ and $y = b$.

Proof. By definition, $f(x, y)$ has a maximum value or a minimum value at $x = a, y = b$ according as $f(a+h, b+k) - f(a, b)$ is negative or positive *i.e.* if $[f(a+h, b+k) - f(a, b)]$ is of invariable sign, for all sufficiently small independent values of h and k, positive or negative.

By Taylor's theorem for two variables, we have

$$f(a+h, b+k) = f(a, b) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{x=a, y=b} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)_{x=a, y=b} + \ldots$$

$$\therefore \quad f(a+h, b+k) - f(a, b) = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{x=a, y=b} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)_{x=a, y=b} + \ldots \quad ...(1)$$

By taking h and k sufficiently small, the first degree terms in h and k can be made to govern the sign of right hand side and therefore of the left hand side of (1). Now h and k are both positive as well as negative thus on changing the sign of h and k the sign of right hand and therefore of left hand side changes. But from the definition of a maximum or minimum value, for all positive and negative arbitrary small values of h and k

$$f(a+h, b+k) - f(a, b)$$

must be of invariable sign. Thus the necessary condition for the existence of a maxima or minima is that the first degree terms of h and k in (1) must vanish *i.e.*,

$$\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{x=a, y=b} = 0. \quad ...(2)$$

Since h and k are independent and non-zero, therefore (2) implies that

$$\left(\frac{\partial f}{\partial x}\right)_{(a, b)} = 0 \text{ and } \left(\frac{\partial f}{\partial y}\right)_{(a, b)} = 0. \quad ...(3)$$

Hence the necessary conditions for $f(x, y)$ to have a maximum or a minimum value at (a, b) are that the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ vanish for $x = a$ and $y = b$.

Note. The conditions obtained above are necessary but not sufficient for the existence of extreme value.

§ 1.4-3. AN ELEMENTARY ALGEBRAIC LEMMA

Let
$$I_2 \equiv ax^2 + 2hxy + by^2$$
$$= \frac{1}{a}[a^2x^2 + 2haxy + aby^2] = \frac{1}{a}[(ax+hy)^2 + (ab-h^2)y^2].$$

The expression within square brackets will be positive if $(ab - h^2)$ is positive and in that case the sign of expression I_2 will be the same as that of a *i.e.*, I_2 will be

(*i*) positive if $(ab - h^2)$ is positive and a is positive;

(*ii*) negative if $(ab - h^2)$ is positive and a is negative.

In case $(ab - h^2)$ is not positive, we can say nothing about the sign of the expression in square brackets and hence nothing about the sign of the given quadratic expression I_2.

§ 1.4-4. SUFFICIENT CONDITIONS FOR MAXIMA AND MINIMA (LAGRANGE'S CONDITION FOR TWO INDEPENDENT VARIABLES)

Theorem. *Let* $\left(\frac{\partial f}{\partial x}\right)_{(a, b)} = 0$ *and* $\left(\frac{\partial f}{\partial y}\right)_{(a, b)} = 0$

Let $\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a, b)} = r$, $\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a, b)} = s$ *and* $\left(\frac{\partial^2 f}{\partial y^2}\right)_{(a, b)} = t$. *Then*

(*i*) *if $(rt - s^2) > 0$ and $r > 0$, $f(x, y)$ is minimum at (a, b);*

(*ii*) *if $(rt - s^2) > 0$ and $r < 0$, $f(x, y)$ is maximum at (a, b);*

(*iii*) *if $(rt - s^2) < 0$, $f(x, y)$ is neither maximum nor minimum at (a, b);*

(*iv*) *if $(rt - s^2) = 0$ the case is doubtful.*

Proof. By Taylor's theorem for two variables, we have

$$f(a+h, b+k) - f(a, b) = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{x=a, y=b} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)_{x=a, y=b} + \ldots$$

If the necessary condtions are satisfied *i.e.*, $\left(\frac{\partial f}{\partial x}\right)_{(a, b)} = 0$, $\left(\frac{\partial f}{\partial y}\right)_{(a, b)} = 0$, then

$$f(a+h, b+k) - f(a, b) = \frac{1}{2}(rh^2 + 2shk + tk^2) + R_3 \quad ...(1)$$

where R_3 consists of terms of third and higher orders in h and k. For sufficiently small values of h and k the sign of R.H.S. and hence of L.H.S. of (1) is governed by the second degree terms *i.e.*, by the expression $(rh^2 + 2shk + tk^2)$.

Now $\quad rh^2+2shk+tk^2=\frac{1}{r}[r^2h^2+2rshk+rtk^2]=\frac{1}{r}[(r^2h^2+2rhsk+s^2k^2)+rtk^2-s^2k^2]$

$$=\frac{1}{r}[(rh+sk)^2+(rt-s^2)k^2]. \quad ...(2)$$

Now following cases arise :

Case I. When $(rt-s^2)>0$ and $r>0$.

In this case, the R.H.S. of (2) is clearly positive and therefore, the R.H.S. of (1) is positive. And so the L.H.S. of (1) is positive *i.e.*, $[f(a+h, b+k)-f(a, b)]>0$ for all values of h and k.

Hence, in this case, $f(x, y)$ has a minima at (a, b).

Case II. When $(rt-s^2)>0$ and $r<0$.

In this case, the R.H.S. of (2) is clearly negative and therefore the R.H.S. of (1) is negative. And so the L.H.S. of (1) is negative *i.e.*, $[f(a+h, b+k)-f(a, b)]<0$ for all values of h and k.

Hence, in this case, $f(x, y)$ has a maxima at (a, b).

Case III. When $(rt-s^2)<0$.

In this case, we can say nothing about the sign of the second degree terms in the R.H.S. (1) *i.e.*, $(rh^2+2skh+tk^2)$ is not of invariable sign.

Hence in this case, we have neither a maxima nor a minima at $x=a$, $y=b$.

Case IV. When $(rt-s^2)=0$ *i.e.*, $rt=s^2$ and $r\neq 0$.

In this case, (2) becomes

$$rh^2+2shk+tk^2=\frac{1}{r}(rh+ks)^2$$

and so it has the same sign as r. Hence maximum if $r<0$ and minimum if $r>0$.

In case, if $rt=s^2$ and $\frac{h}{k}=-\frac{s}{r}$ (*i.e.*, $rh+sk=0$), then second degree terms vanish and we have to consider terms of higher order in (1).

The case is, therefore doubtful.

In case $r=0$, then from the condition $rt-s^2=0$, we must have $s=0$. Therefore

$$rh^2+2shk+tk^2=tk^2$$

which is clearly zero when $k=0$ for all values of h. Hence the case is again doubtful.

Hence if $rt-s^2=0$, the case is doubtful and further investigation is required to decide whether the function $f(x, y)$ is a maximum or a minimum at (a, b).

§ 1.4-5. WORKING RULE FOR MAXIMA AND MINIMA OF FUNCTION OF TWO VARIABLES

Let $f(x, y)$ be a given function.

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

2. Solve the equations $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ for x and y. Then pairs of values of x and y, thus obtained will give stationary values of $f(x, y)$. Let (a, b) be one of these pairs.

3. Find $r=\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)}$; $s=\left(\frac{\partial^2 f}{\partial x\partial y}\right)_{(a,b)}$ and $t=\left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)}$ and calculate $rt-s^2$.

4. (*i*) if $(rt-s^2)$ is positive and $r>0$, then $f(x, y)$ has a minimum at (a, b);
(*ii*) if $rt-s^2>0$ and $r<0$ then $f(x, y)$ has a maximum at (a, b);
(*iii*) if $(rt-s^2)$ is negative then $f(x, y)$ has neither maximum nor minimum at (a, b).
(*iv*) if $rt-s^2=0$, then the case is doubtful and further investigation is required.

ILLUSTRATIVE EXAMPLES

Ex. 1. *Discuss the maximum or minimum values of the function* $f(x, y)=x^3-4xy+2y^2$.

Sol. Here $\frac{\partial f}{\partial x}=3x^2-4y$ and $\frac{\partial f}{\partial y}=-4x+4y$. For maxima or minima of f, we have

$$\frac{\partial f}{\partial x}=0,\ \frac{\partial f}{\partial y}=0 \quad \Rightarrow \quad 3x^2-4y=0, -4x+4y=0$$

$$\Rightarrow \quad 3x^2-4y=0, x=y \quad \Rightarrow \quad 3y^2-4y=0, x=y$$

$$\Rightarrow \quad y(3y-4)=0, x=y \quad \Rightarrow \quad y=0 \text{ or } y=\frac{4}{3}, x=y$$

$$\Rightarrow \quad (0, 0) \text{ and } \left(\frac{4}{3}, \frac{4}{3}\right) \text{ are critical points.}$$

Now $\quad r=\frac{\partial^2 f}{\partial x^2}=6x,\ s=\frac{\partial^2 f}{\partial x\partial y}=-4,\ t=\frac{\partial^2 f}{\partial y^2}=4.$

Thus, at (0, 0) : $\quad r=0,\ rt-s^2=(0)(4)-(-4)^2=-16<0.$

Hence we conclude that (0, 0) is a saddle point of f. Again at $\left(\frac{4}{3}, \frac{4}{3}\right)$:

$$r=8, s=-4, t=4.$$

Since $\quad r=8>0$ and $rt-s^2=8(4)-16=16>0.$

Hence function f has a minimum value at $x=y=4/3$. Thus, at $x=y=4/3$,

$$\text{the minimum value of } f=f\left(\frac{4}{3}, \frac{4}{3}\right)=-\frac{32}{27}$$

Ex. 2. *Discuss the maximum or minimum values of u in the following cases* :

(*i*) $u=xy+a^3\left(\frac{1}{x}+\frac{1}{y}\right)$. $\qquad$ (*ii*) $u=x^3-y^2-3x$.

Sol. (i) Here $\quad u=xy+a^3\left(\frac{1}{x}+\frac{1}{y}\right)$.

$$\therefore \quad \frac{\partial u}{\partial x}=y-\frac{a^3}{x^2} \text{ and } \frac{\partial u}{\partial y}=x-\frac{a^3}{y^2}$$

Then for maxima or minima of u, we have

$$\frac{\partial u}{\partial x}=0, \frac{\partial u}{\partial y}=0 \Rightarrow y-\frac{a^3}{x^2}=0, x-\frac{a^3}{y^2}=0$$

$$\Rightarrow \quad a^3=x^2y, a^3=xy^2 \quad \Rightarrow \quad a^3=x^2y=xy^2 \quad \Rightarrow \quad x=y=a.$$

Now $\quad r=\frac{\partial^2 u}{\partial x^2}=\frac{2a^3}{x^3},\ s=\frac{\partial^2 u}{\partial x\partial y}=1$ and $t=\frac{\partial^2 u}{\partial y^2}=\frac{2a^3}{y^3}.$

Thus at $x=y=a$; we get

$$r=2, s=1 \text{ and } t=2$$

$$\therefore \quad rt-s^2=2.2.-1^2=3>0.$$

Since r and $rt-s^2$ are both positive, u is minimum when $x=y=a$.

Hence the minimum value of the function is $a^2+a^2+a^2=3a^2$.

(ii) Here $u=x^3-y^2-3x$. Hence

$$\frac{\partial u}{\partial x}=3x^2-3 \text{ and } \frac{\partial u}{\partial y}=-2y.$$

For maxima or minima of u, we have

$$\frac{\partial u}{\partial x}=0,\ \frac{\partial u}{\partial y}=0 \Rightarrow 3x^2-3=0,\ -2y=0 \quad \Rightarrow \quad x=\pm 1,\ y=0.$$

Hence the critical points are (1, 0) and (– 1, 0).

Now $$r=\frac{\partial^2 u}{\partial x^2}=6x,\ s=\frac{\partial^2 u}{\partial x\,\partial y}=0 \quad \text{and} \quad t=\frac{\partial^2 u}{\partial y^2}=-2.$$

Thus at (– 1, 0), we get

$$r=-6,\ s=0 \text{ and } t=-2.$$

$\therefore$ $$rt-s^2=(-6).(-1)-0=12>0.$$

Since $r<0$ and $rt-s^2>0$, u is maximum at (– 1, 0). Hence the maximum value of the function

$$=-1-0+3=2.$$

Now at the point (1, 0), we get

$$r=6(1)=6,\ s=0 \text{ and } t=-2.$$

If gives $rt-s^2=6(-2)-0=-12<0.$

Therefore u is neither maximum nor minimum at (1, 0). It is a saddle point.

Ex. 3. *Discuss the maxima and minima of the function* $ax^3y^2-x^4y^2-x^3y^3$.

Sol. Let $$u=ax^3y^2-x^4y^2-x^3y^3.$$

$\therefore$ $$\frac{\partial u}{\partial x}=3ax^2y^2-4x^3y^2-3x^2y^3 \quad \text{and} \quad \frac{\partial u}{\partial y}=2ax^3y-2x^4y-3x^3y^2.$$

For maxima or minima of u, we have

$$\frac{\partial u}{\partial x}=0,\ \frac{\partial u}{\partial y}=0 \Rightarrow 3a-4x-3y=0,$$

$$2a-2x-3y=0 \quad \Rightarrow \quad x=\frac{a}{2},\ y=\frac{a}{3}$$

$\therefore$ The point $\left(\frac{a}{2},\frac{a}{3}\right)$ is a critical point.

Now at the point $\left(\frac{a}{2},\frac{a}{3}\right)$, we have

$$r=\frac{\partial^2 u}{\partial x^2}=6axy^2-12x^2y^2-6xy^3=6xy^2(a-2x-y)=6.\frac{a}{2}.\frac{a^2}{9}.\left(a-a-\frac{a}{3}\right)=-\frac{a^4}{9},$$

$$s=\frac{\partial^2 u}{\partial x\partial y}=6ax^2y-8x^3y-9x^2y^2=x^2y(6a-8x-9y)=\frac{a^2}{4}.\frac{a}{3}.(6a-4a-3a)=-\frac{a^4}{12},$$

$$t=\frac{\partial^2 u}{\partial y^2}=2ax^3-2x^4-6x^3y=2x^3(a-x-3y)=2.\frac{a^3}{8}.\left(a-\frac{a}{2}-a\right)=-\frac{a^4}{8}.$$

Now $$rt-s^2=\left(\frac{1}{72}-\frac{1}{144}\right)a^8>0.$$

Since r is –ve and $rt-s^2$ is +ve, therefore u is maximum when $x=\frac{a}{2},\ y=\frac{a}{3}.$

Ex. 4. *Discuss the maxima and minima of the function* x^3+y^3-3axy.

[R.G.T.U. Dec. 2003, June 2004, Feb. 2005]

Sol. Let x^3+y^3-3axy.

$\therefore$ $$\frac{\partial u}{\partial x}=3x^2-3ay \quad \text{and} \quad \frac{\partial u}{\partial y}=3y^2-3ax.$$

Thus for maxima or minima of u

$$\frac{\partial u}{\partial x}=0,\ \frac{\partial u}{\partial y}=0 \quad \Rightarrow \quad 3x^2-3ax=0,\ 3y^2-3ax=0$$

$\Rightarrow$ $$x^2=ay,\ y^2=ax \quad \Rightarrow \quad x=y=a.$$

$\therefore$ The point (a, a) is a critical point.

Now $$r=\frac{\partial^2 u}{\partial x^2}=6x,\ s=\frac{\partial^2 u}{\partial x\partial y}=-3a \quad \text{and} \quad t=\frac{\partial^2 u}{\partial y^2}=6y.$$

At the point (a, a), we have

$$r=6a,\ s=-3a \text{ and } t=6a.$$

If gives $$rt-s^2=(6a)(6a)-(-3a)^2=27a^2>0.$$

Since $rt-s^2$ is +ve and r is +ve or –ve according as a is +ve or –ve, we have a maximum or minimum according as a is – ve or + ve at $x=y=a$.

Ex. 5. *Discuss the maxima and minima of the function* $x^2y^2-5x^2-5y^2-8xy$.

Sol. Let $u=x^2y^2-5x^2-8xy-5y^2$. For maxima or minima of u, we have

$$\frac{\partial u}{\partial x}=2xy^2-10x-8y=0 \qquad \text{...(1)}$$

and $$\frac{\partial u}{\partial y}=2x^2y-8x-10y=0. \qquad \text{...(2)}$$

Subtracting (2) from (1), we get

$$2xy(y-x)+2(y-x)=0 \quad \Rightarrow \quad (y-x)(xy+1)=0$$

$\Rightarrow$ either $y=x$ or $y=-\frac{1}{x}.$

When $y=x$, then (1) $\Rightarrow 2x^3-18x=0 \Rightarrow x=0,\pm 3.$

And when $y=-\frac{1}{x}$,

(1) $\Rightarrow \frac{2}{x}-10x+\frac{8}{x}=0 \Rightarrow 1-x^2=0 \Rightarrow x\pm 1.$

Hence the solutions are

$$x=y=0;\ x=y=3;\ x=y=-3;\ x=1,\ y=-1;\ x=-1,\ y=1.$$

Thus the critical points are (0, 0), (3, 3), (– 3, 3), (1, – 1) and (– 1, 1).

Now $$r=\frac{\partial^2 u}{\partial x^2}=2y^2-10,\ s=\frac{\partial^2 u}{\partial x\partial y}=4xy-8 \quad \text{and} \quad t=\frac{\partial^2 u}{\partial y^2}=2x^2-10.$$

At $x=y=0$, we have

$$r=-10,\ s=-8,\ t=-10.$$

So $$rt-s^2=100-64=36>0.$$

Since r is –ve and $rt-s^2$ is +ve, we have a maximum when

$$x=y=0.$$

At $x=y=\pm 3$, we have

$$r=8,\ s=28,\ t=8.$$

$\therefore$ $rt-s^2=64-28\times 28=a$ negative quantity.

Since $rt-s^2$ is –ve, we have neither a maximum nor a minimum when $x=y=\pm 3$.

Again when $x=\pm 1,\ y=\mp 1$, we have

$$r=-8,\ s=-4-8=-12,\ t=-8.$$

$\therefore$ $rt-s^2=64-144=a$ negative quantity.

Hence in this case also we have neither a maximum nor a minimum.

Ex. 6. *Investigate the maxima and minima of* $u=2(x-y)^2-x^4-y^4$ *leaving aside any doubtful case that may arise.*

Sol. For maxima or minima of u, we have

$$\frac{\partial u}{\partial x}=4(x-y)-4x^3=0$$

and

$$\frac{\partial u}{\partial y}=-4(x-y)-4y^3=0.$$

Adding (1) and (2), we have

$$x^3+y^3=0 \Rightarrow y=-x \Rightarrow 4.(2x)-4x^3=0 \quad \text{[using (1)]}$$

$$\Rightarrow \quad 2x-x^3=0 \Rightarrow x=0 \text{ or } x=0\pm\sqrt{2}$$

$\therefore \quad y=0$ or $y=\mp\sqrt{2}$.

Thus we have to consider the three points $(0,0)$; $(\sqrt{2},-\sqrt{2})$ and $(-\sqrt{2},\sqrt{2})$.

Now $$r=\frac{\partial^2 u}{\partial x^2}=4-12x^2,\ s=\frac{\partial^2 u}{\partial x\partial y}=-4,\ t=\frac{\partial^2 u}{\partial y^2}=4-12y^2.$$

At $(0,0)$: $r=4, s=-4, t=4, rt-s^2=0$.

$\therefore$ Doubtful case at $(0,0)$.

At $(\sqrt{2},-\sqrt{2})$: $r=-20, s=-4, t=-20$,

$$rt-s^2=(-20).(20)-(-4)^2=400-16=+\text{ve}.$$

Also $$r=-\text{ve}$$

Hence there is a maximum at $(\sqrt{2},-\sqrt{2})$.

At the point $(\sqrt{2},-\sqrt{2})$: $r=-20, s=-4, t=-20, rt-s^2=+$ ve. Also $r=-$ ve.

Hence there is a maximum at $(-\sqrt{2},\sqrt{2})$ also.

Ex. 7. *If $f(x,y)=2x^4+y^4-2x^2-2y^2$, then show that f has a maximum at $(0, 0)$ and a minimum at $\left(\frac{1}{\sqrt{2}},1\right)$.*

Sol. For maxima or minima of f,

$$\frac{\partial f}{\partial x}=8x^3-4x=0 \quad ...(1)$$

$$\frac{\partial f}{\partial y}=4y^3-4y=0. \quad ...(2)$$

Solving (1) and (2), we get

$$x=0 \text{ or } x=\pm\frac{1}{\sqrt{2}} \text{ and } y=0 \text{ or } y=\pm 1.$$

Now $$r=\frac{\partial^2 f}{\partial x^2}=24x^2-4,\ s=\frac{\partial^2 f}{\partial x\partial y}=0,\ t=\frac{\partial^2 f}{\partial y^2}=12y^2-4.$$

(i) At $(0,0)$: $r=-4, s=0, t=-4$ and $rt-s^2>0$. Also r is negative. Hence f has a maximum at $(0,0)$.

(ii) At $\left(\frac{1}{\sqrt{2}},1\right)$: $r=8, s=0, t=8$ and $rt-s^2>0$. Also $r=+$ve. Hence f has a mininmm at $\left(\frac{1}{\sqrt{2}},1\right)$.

Ex. 8. *Discuss the maximum or minimum value of $u=x^3y^2(1-x-y)$.*

[R.G.T.U. Dec. 2002, Jan./Feb. 2007]

Sol. For maxima or a minima of u, we have

$$\frac{\partial u}{\partial x}=0 \Rightarrow 3x^2y^2-4x^3y^2-3x^2y^3=0 \Rightarrow x^2y^2(3-4x-3y)=0 \quad ...(1)$$

$$\frac{\partial u}{\partial y}=0 \Rightarrow 2x^3y-2x^4y-3x^3y^2=0 \Rightarrow x^3y(2-2x-3y)=0. \quad ...(2)$$

On solving, we get

$$x=\frac{1}{2},\ y=\frac{1}{3}.$$

Now $$r=\frac{\partial^2 u}{\partial x^2}=6xy^2-12x^2y^2-6xy^3=-\frac{1}{9} \text{ at } \left(\frac{1}{2},\frac{1}{3}\right)$$

$$s=\frac{\partial^2 u}{\partial x\partial y}=6x^2y-8x^3y-9x^2y^2=-\frac{1}{12} \text{ at } \left(\frac{1}{2},\frac{1}{3}\right)$$

$$t=\frac{\partial^2 u}{\partial y^2}=2x^3-2x^4-6x^3y=-\frac{1}{8} \text{ at } \left(\frac{1}{2},\frac{1}{3}\right).$$

$$\therefore \quad rt-s^2=\frac{1}{72}-\frac{1}{144}=\frac{1}{72}=+\text{ve}.$$

Since $r<0$ and $rt-s^2>0$ therefore u has a maximum at $\left(\frac{1}{2},\frac{1}{3}\right)$.

$$\therefore \quad \text{maximum value of } u=\left(\frac{1}{2}\right)^3\cdot\left(\frac{1}{3}\right)^2\cdot\left(1-\frac{1}{2}-\frac{1}{3}\right)=\frac{1}{72}\left(\frac{6-3-2}{6}\right)=\frac{1}{432}.$$

Ex. 9. *Discuss the maximum and minimum of u, where $u=2a^2xy-3ax^2y-ay^3+x^3y+xy^3$.*

Sol. For maxima or minima of u, we have

$$\frac{\partial u}{\partial x}=2a^2y-6axy+3x^2y+y^3=0$$

$$\Rightarrow \quad 2a^2-6ax+3x^2+y^2=0 \quad ...(1)$$

$$\frac{\partial u}{\partial y}=2a^2x-3ax^2-3ay^2+x^3+3xy^2. \quad ...(2)$$

Multiplying (1) by x and subtracting from (2), we get

$$3ax^2-3ay^2-2x^3+2xy^2=0 \Rightarrow (x^2-y^2)(3a-2x)=0$$

$$\therefore \quad x=\frac{3a}{2} \text{ or } x=\pm y.$$

If $x=\frac{3a}{2}$ then $y=\pm\frac{a}{2}$.

If $x=y$ then $y=a, a/2$.

If $x=-y$ then $y=-a,-a/2$. ...(1)

Thus the stationary points are

$$\left(\frac{3a}{2},\frac{a}{2}\right),\left(\frac{3a}{2},-\frac{a}{2}\right),(a,a),\left(\frac{a}{2},\frac{a}{2}\right),(a,-a)\left(\frac{a}{2},-\frac{a}{2}\right).$$

Now $$r=\frac{\partial^2 u}{\partial x^2}=-6ay+6xy$$

$$s=\frac{\partial^2 u}{\partial x\partial y}=2a^2-6ax+3x^2+3y^2$$

$$t=\frac{\partial^2 u}{\partial y^2}=-6ay+6xy.$$

Now see following table :

stationary points	$\left(\frac{3a}{2},\frac{a}{2}\right)$	$\left(\frac{3a}{2},-\frac{a}{2}\right)$	(a,a)	$\left(\frac{a}{2},\frac{a}{2}\right)$	$(a,-a)$	$\left(\frac{a}{2},-\frac{a}{2}\right)$
$r=$	$3a^2/2$	$-3a^2/2$	0	$-3a^2/2$	0	$3a^2/2$
$s=$	$a^2/2$	$a^2/2$	$2a^2$	$a^2/2$	$2a^2$	$a^2/2$
$t=$	$3a^2/2$	$-3a^2/2$	0	$-3a^2/2$	0	$3a^2/2$
$rt-s^2=$	+ ve	+ ve	− ve	+ ve	− ve	+ ve

Hence u is maximum at $(3a/2, -a/2)$ and $(a/2, a/2)$ and minimum at $(3a/2, a/2)$ and $(a/2, -a/2)$
There is neither maximum nor minimum at (a, a) and $(a, -a)$.

Ex. 10. *Find all the maximum and minimum values of functions f given by*

(i) $f(x, y) = xy(a - x - y)$.

(ii) $f(x, y) = x^3 + y^3 - 63(x + y) + 12xy$.

Sol. (i) For maxima and minima of f, we have

$$\frac{\partial f}{\partial x} = ay - 2xy - y^2 = 0 \quad ...(1)$$

$$\frac{\partial f}{\partial y} = ax - x^2 - 2xy = 0. \quad ...(2)$$

Subtraction (2) from (1), we have

$$a(y - x) - (y^2 - x^2) = 0 \Rightarrow (y - x)(a - y - x) = 0 \Rightarrow y = x \text{ or } y + x = a$$

Putting $y = x$ in (1), we have

$$ya - 2y^2 - y^2 = 0 \Rightarrow 3y^2 - ya = 0 \Rightarrow y = 0, y = a/3.$$

When $y = a - x$, (2)

$$\Rightarrow xa - x^2 - 2x(a - x) = 0 \Rightarrow xa - x^2 - 2ax + 2x^2 = 0$$

$$\Rightarrow x^2 - ax = 0 \Rightarrow x = 0, x = a.$$

Thus f may have maxima or minima at $(0, 0)$; $(a/3, a/3)$; $(0, a)$; $(a, 0)$.

Now

$$r = \frac{\partial^2 f}{\partial x^2} = -2y = 0; -\frac{2a}{3}; -2a; 0$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y = a; -\frac{a}{3}; -a; -a$$

$$t = \frac{\partial^2 f}{\partial y^2} = -2x; 0; -\frac{2a}{3}; 0; -2a$$

$$rt - s^2 = -\text{ve}; +\text{ve}; -\text{ve}; -\text{ve}.$$

Thus f has a maximum at $(a/3, a/3)$ and has no extreme value at other points.

$\therefore$ maximum value of $f = \frac{a^2}{9}\left(a - \frac{a}{3} - \frac{a}{3}\right) = \frac{a^3}{27}$

(ii) For maxima or minima of f, we have

$$\frac{\partial f}{\partial x} = 3x^2 - 63 + 12y = 0. \quad ...(1)$$

$$\frac{\partial f}{\partial y} = 3y^2 - 63 + 12x = 0. \quad ...(2)$$

Subtracting (2) from (1), we have

$$3(x^2 - y^2) + 12(y - x) = 0$$

$$\Rightarrow 3(x - y)(x + y) - 12(x - y) = 0$$

$$\Rightarrow 3(x - y)(x + y - 4) = 0 \Rightarrow x = y \text{ or } x + y = 4.$$

(i) Putting $x = y$ in (1), we get

$$3x^2 + 12x - 6 = 0 \Rightarrow x^2 + 4x - 21 = 0$$

$$\Rightarrow (x + 7)(x - 3) = 0 \Rightarrow x = -7 \text{ or } 3.$$

Thus f may have maxima or minima at $(-7, -7)$ and $(3, 3)$.

(ii) Putting $x = 4 - y$ in (2) we get

$$3y^2 - 63 + 12(4 - y) = 0 \Rightarrow 3y^2 - 12y - 15 = 0$$

$$\Rightarrow y^2 - 4y - 5 = 0 \Rightarrow (y - 5)(y + 1) = 0$$

$$\Rightarrow y = 5, -1 \Rightarrow y = 5, -1; x = -1, 5. \quad [\because x = 4 - y]$$

Thus f may have maxima or minima at $(-1, 5)$ and $(5, -1)$.

Now $r = 6x, s = 12, t = 6y$.

Points	(−7, −7)	(3, 3)	(−1, 5)	(5, −1)
$r =$	−42	18	−6	30
$s =$	12	12	12	12
$t =$	−42	18	30	−6
$rt - s^2$	+ve	+ve	−ve	−ve
Result	Maximum	Minimum	neither Maxima nor Minima	neither Maxima nor Minima

Hence f has a maximum at $(-7, -7)$ and a minima at $(3, 3)$.

$\therefore$ Maximum value of $f = (-7)^3 + (-7)^3 - 63(-7 - 7) + 12(-7)(-7) = 784$

and minimum value of $f = (3)^3 + (3)^3 - 63 \times 6 + 12 \times 3 \times 3 = -216$.

Ex. 11. *Find the maxima and minima of the function $u = \sin x \sin y \sin(x + y)$.*

Sol. For maxima or minima of u, we have

$$\frac{\partial u}{\partial x} = \cos x \sin y \sin(x+y) + \sin x \sin y \cos(x+y) = \sin y \sin(2x + y) = 0 \quad ...(1)$$

$$\frac{\partial u}{\partial y} = \sin x \cos y \sin(x + y) + \sin x \sin y \cos(x + y) = \sin x \sin(2y + x) = 0. \quad ...(2)$$

The given function u is periodic with period π, both for x and y, therefore it is sufficient to consider the values of x and y between 0 and π.

Now (1) $\Rightarrow \sin y = 0$ or $\sin(2x + y) = 0$

$\Rightarrow y = 0$ or $2x + y = \pi$ or 2π ...(3)

and (2) $\Rightarrow x = 0$ or $2y + x = \pi$ or 2π. ...(4)

Again solving $2x + y = \pi$ or 2π with equation (4), we have $(\pi/3, \pi/3)$, $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$.

Thus the critical points are $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$.

Now $r = \frac{\partial^2 u}{\partial x^2} = 2 \sin y \cos(2x + y)$, $s = \frac{\partial^2 u}{\partial x \partial y} = \sin(2x + 2y)$ and $t = \frac{\partial^2 u}{\partial y^2} = 2 \sin x \cos(x + 2y)$.

At $(\pi/3, \pi/3)$ *i.e.*, when $x = y = \pi/3$, we have

$$r = 2 . (\sqrt{3}/2) . (-1) = -\sqrt{3}, \quad s = \sin(4\pi/3) = -(\sqrt{3}/2) \text{ and } t = 2 . (\sqrt{3}/2) . (-1) = -\sqrt{3}$$

$\therefore$ $rt - s^2 = 3 - (3/4) = 9/4 > 0$ and $r = -\sqrt{3} < 0$.

Hence by Lagrange's condition u is maximum at $(\pi/3, \pi/3)$ *i.e.*, at

$$x = y = \frac{1}{3}\pi$$

and $u_{max} = \sin(\pi/3) \sin(\pi/3) . \sin(2\pi/3) = 3\sqrt{3}/8$.

At $(2\pi/3, 2\pi/3)$ *i.e.*, when $x = y = 2\pi/3$, we have

$$r = 2 . (\sqrt{3}/2) . (1) = \sqrt{3} = t, \; s = \sqrt{3}/2$$

$\therefore$ $rt - s^2 = 3 - (3/4) = 9/4 > 0$ and $r = \sqrt{3} > 0$.

Hence by Lagrange's condition u is minimum at $x = y = 2\pi/3$.

$$u_{min} = \sin(2\pi/3) \sin(2\pi/3) \sin(4\pi/3) = -3\sqrt{3}/8.$$

Ex. 12. *Discuss the maxima and minima of $u = \sin x + \sin y + \sin(x + y)$.*

Sol. For maxima or minima of u, we have

$$\frac{\partial u}{\partial x} = \cos x + \cos(x + y) = 0 \quad ...(1)$$

and

$$\frac{\partial u}{\partial x} = \cos y + \cos(x + y) = 0 \quad ...(2)$$

(1) and (2) $\Rightarrow \cos x = \cos y \Rightarrow x = y \Rightarrow \cos x + \cos (x + x) = 0$

$\Rightarrow \cos x + 2\cos^2 x - 1 = 0 \Rightarrow 2\cos^2 x + 2\cos x - \cos x - 1 = 0$

$\Rightarrow 2\cos x(\cos x + 1) - 1(\cos x + 1) = 0$

$\Rightarrow (2\cos x - 1)(\cos x + 1) = 0 \Rightarrow 2\cos x - 1 = 0$ or $\cos x + 1 = 0$

$\Rightarrow \cos x = \frac{1}{2} = \cos \pi/3$ or $\cos x = -1 = \cos \pi \Rightarrow x = \pi/3$ or $x = \pi$.

Now

$$r = \frac{\partial^2 u}{\partial x^2} = -\sin x - \sin (x+y)$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = -\sin (x+y) \quad \text{and} \quad t = \frac{\partial^2 u}{\partial y^2} = -\sin y - \sin (x+y).$$

When $x = y = \pi/3$, we have

$$r = -\sin \frac{\pi}{3} - \sin\left(\frac{2\pi}{3}\right) = -\sqrt{3}$$

$$s = -\sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

and

$$t = -\sin \frac{\pi}{3} - \sin\left(\frac{2\pi}{3}\right) = -\sqrt{3}.$$

$\therefore$

$$rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{9}{4} = +\text{ve}.$$

Since $rt - s^2$ is +ve and r is −ve, hence we have a maxima at $x = y = \pi/3$.

When $x = y = \pi$, we have

$$r = 0,\ s = 0,\ t = 0$$

$\therefore$ $rt - s^2 = 0$.

This case is doubtful and hence we shall leave it.

Ex. 13. *Find the maxima and minima of the function* $u = 2\sin \frac{x+y}{2} \cos \frac{x+y}{2} + \cos (x+y)$.

Sol.

$$u = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x+y}{2}\right) + \cos (x+y)$$

or

$$u = \sin x + \sin y + \cos (x+y).$$

For maxima or minima of u, we have

$$\frac{\partial u}{\partial x} = \cos x - \sin (x+y) = 0$$

and

$$\frac{\partial u}{\partial y} = \cos y - \sin (x+y) = 0.$$

Now (1) and (2) $\Rightarrow \cos x = \cos y \Rightarrow x = y$.

Putting $x = y$ is (1), we have

$$\cos x - \sin 2x = 0 \Rightarrow \cos x - 2\sin x \cos x = 0$$

$\Rightarrow$ $\cos x(1 - 2\sin x) = 0 \quad \Rightarrow \quad \cos x = 0 = \cos \pi/2$

or $\sin x = \frac{1}{2} = \sin \pi/6 \quad \Rightarrow x = 2n\pi \pm \pi/2$ and $x = n\pi + (-1)^n (\pi/6)$.

Now

$$r = \frac{\partial^2 u}{\partial x^2} = -\sin x - \cos (x+y),\ s = \frac{\partial^2 u}{\partial y \partial x} = -\cos (x+y)$$

and

$$t = \frac{\partial^2 u}{\partial y^2} = -\sin y - \cos (x+y).$$

When

$$x = y = 2n\pi - \pi/2$$

$$r = -\sin\left(2n\pi - \frac{\pi}{2}\right) - \cos (4n\pi - \pi) = \sin (\pi/2) - \cos \pi = 2$$

$$s = -\cos (4n\pi - \pi) = -\cos \pi = 1,\ t = 2$$

$$rt - s^2 = 2 \times 2 - 1 = 3 = +\text{ ve}.$$

Hence there is a minimum at $x = y = 2n\pi - \pi/2$.

When $x = y = 2n\pi + \pi/2$

$$r = -\sin (2n\pi + \pi/2) - \cos (4n\pi + \pi) = -\sin \pi/2 - \cos \pi = 0$$

$$s = -\cos (4n\pi + \pi) = -\cos \pi = 1,\ t = 0$$

$\therefore$ $rt - s^2 = 0 - 1 = -1 = -$ ve.

Hence there is neither maximum nor minimum at $x = y = 2n\pi + \pi/2$.

When $x = y = n\pi + (-1)^n . (\pi/6)$

$$r = -\sin \{n\pi + (-1)^n (\pi/6)\} - \cos \{2n\pi + (-1)^n (\pi/3)\}$$

$$= -\sin (\pi/6) - \cos (\pi/3) = -1.$$

Similarly $t = -1$

and $s = -\cos \{2n\pi + (-1)^n (\pi/3)\} = -\cos (\pi/3) = -1/2$.

$\therefore$ $rt - s^2 = (-1)(-1) - (-1/2)^2 = 3/4 = +$ ve.

Hence there is a maxima at $x = y = n\pi + (-1)^n (\pi/6)$.

Ex. 14. *Show that the minimum value of* $(a_1 x + b_1 y + c_1)^2 + (a_2 x + b_2 y + c_2)^2 + \ldots + (a_n x + b_n y + c_n)^2$ *is given by the values of x and y which satisfy the equations :*

$$(\Sigma a_1^2)\, x + (\Sigma a_1 b_1)\, y + (\Sigma a_1 c_1) = 0$$

and

$$(\Sigma a_1 b_1)\, x + (\Sigma b_1^2)\, y + (\Sigma b_1 c_1) = 0.$$

Sol. Let $u = (a_1 x + b_1 y + c_1)^2 + (a_2 x + b_2 y + c_2)^2 + \ldots + (a_n x + b_n y + c_n)^2$.

For a maxima or a minima of u, we have

$$\frac{\partial u}{\partial x} = 2a_1 (a_1 x + b_1 y + c_1) + \ldots + 2a_n (a_n x + b_n y + c_n) = 0$$

$\Rightarrow$ $(\Sigma a_1^2)\, x + (\Sigma a_1 b_1)\, y + (\Sigma a_1 c_1) = 0$ ...(1)

$$\frac{\partial u}{\partial y} = 2b_1 (a_1 x + b_1 y + c_1) + \ldots + 2b_n (a_n x + b_n y + c_n) = 0$$

$\Rightarrow$ $(\Sigma a_1 b_1)\, x + (\Sigma b_1^2)\, y + (\Sigma_1 c_1) = 0.$ ...(2)

Equations (1) and (2) are the required equations.

Also $r = 2\,\Sigma a_1^2,\ s = 2\Sigma a_1 b_1,\ t = 2\Sigma b_1^2$

$$rt - s^2 = 4\,[\Sigma a_1^2\, \Sigma b_1^2 - (\Sigma a_1 b_1)^2]$$

$$= 4\Sigma\,(a_1 b_2 - a_2 b_1)^2 \quad \text{[by Lagrange's Identity]}$$

$$= +\text{ve}.$$

Also r is positive. Hence u is minimum for the values of x and y satisfying the equations (1) and (2).

Ex. 15. *If x, y, z are angles of a triangle, then find the maximum value of sin x sin y sin z.*

Sol. Let $u = \sin x \sin y \sin z$, where

$$x + y + z = \pi.$$

Again, $x + y + z = \pi$

$\Rightarrow$ $x + y = \pi - z$

$\Rightarrow$ $\sin (x+y) = \sin (\pi - z) \quad \Rightarrow \quad \sin (x+y) = \sin z.$

$\therefore$ $u = \sin x \sin y \sin (x+y)$.

Now see Ex. 11 above.

Ex. 16. *Find a point within a triangle such that the sum of the squares of its distances from the three vertices is a minimum.*

PROBLEM SET

Discuss the maximum or minimum values of the following functions :

1. $u = x^4 + 2x^2y - x^2 + 3y^2$.
2. $u = x^2 - 3xy + y^2 + 2x$.
3. $u = x^3y - y^3x - x + y$.
4. $u = x^2 + y^2 + 6x + 12$.
5. $u = 6xy + (47 - x - y)(4x + 3y)$.

6. Prove that the maxima or minima of the fraction $u = \dfrac{ax^2 + by^2 + 2hxy + 2gx + 2fy + c}{a'x^2 + b'y^2 + 2h'xy + 2g'x + 2f'y + c'}$ are given by the roots of the equation $\begin{vmatrix} a - a'u & h - h'u & g - g'u \\ h - h'u & b - b'u & f - f'u \\ g - g'u & f - f'u & c - c'u \end{vmatrix} = 0$.

7. Let $f(x, y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^5$. Show that $f(x, y)$ has neither a maximum nor a minimum at (0, 0).

8. Find all the maxima and minima of the function $f(x, y) = y^2 + x^2y + x^4$.

ANSWERS

1. Minimum at $\left(\pm\frac{1}{2}\sqrt{3}, -\frac{1}{4}\right)$.
2. Neither maximum nor minimum at $x = 4/5$, $y = 6/$
3. Neither maximum nor minimum at (1, 1) and (– 1, 1).
4. Minima at (– 3, 0).
5. Maxima at $x = 21$, $y = 20$, $u_{max} = 3384$.
8. Minimum at (0, 0).

§ 1.4-6. LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

Let
$$u = \phi(x_1, x_2, \dots, x_n) \quad \dots(1)$$
be a function of n variables $x_1, x_2, x_3, \dots, x_n$. Let these variables be connected by m equations
$$f_1(x_1, x_2, \dots, x_n) = 0,$$
$$f_2(x_1, x_2, \dots, x_n) = 0, \quad \dots(2)$$
$$\dots \quad \dots \quad \dots$$
$$f_m(x_1, x_2, \dots, x_n) = 0.$$

So that there are only $n - m$ independent variables.

For a maximum or minimum of u, we have
$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \frac{\partial u}{\partial x_3} dx_3 + \dots + \frac{\partial u}{\partial x_n} dx_n = 0. \quad \dots(3)$$

Also
$$\left.\begin{aligned} df_1 &= \frac{\partial f_1}{\partial x_1} dx_1 + \frac{\partial f_1}{\partial x_2} dx_2 + \frac{\partial f_1}{\partial x_3} dx_3 + \dots + \frac{\partial f_1}{\partial x_n} dx_n = 0, \\ df_2 &= \frac{\partial f_2}{\partial x_1} dx_1 + \frac{\partial f_2}{\partial x_2} dx_2 + \frac{\partial f_2}{\partial x_3} dx_3 + \dots + \frac{\partial f_2}{\partial x_n} dx_n = 0, \\ \dots &= \dots \quad \dots \quad \dots \quad \dots = 0 \\ df_m &= \frac{\partial f_m}{\partial x_1} dx_1 + \frac{\partial f_m}{\partial x_2} dx_2 + \frac{\partial f_m}{\partial x_3} dx_3 + \dots + \frac{\partial f_m}{\partial x_n} dx_n = 0 \end{aligned}\right\} \quad \dots(4)$$

Multiplying (3) by 1 and (4) by $\lambda_1, \lambda_2, \dots, \lambda_m$ respectively and adding we get a result which may written as
$$P_1 dx_1 + P_2 dx_2 + P_3 dx_3 + \dots + P_n dx_n = 0$$
where
$$P_r = \frac{\partial u}{\partial x_r} + \lambda_1 \frac{\partial f_1}{\partial x_r} + \lambda_2 \frac{\partial f_2}{\partial x_r} + \dots + \lambda_m \frac{\partial f_m}{\partial x_r}.$$

Now the m quantities $\lambda_1, \lambda_2, \dots, \lambda_m$ are at our choice. We choose them to satisfy the m linear equations
$$P_1 = P_2 = P_3 = \dots = P_m = 0. \quad \dots(6)$$

With the help of (6), equation (5) now reduces to
$$P_{m+1} dx_{m+1} + P_{m+2} dx_{m+2} + \dots + P_n dx_n = 0. \quad \dots(7)$$

It is immaterial which of the $n - m$ of the n variables we regarded as independent. Let them be $x_{m+1}, x_{m+2}, \dots, x_n$. Then since the $n - m$ quantities $dx_{m+1}, dx_{m+2}, \dots, dx_n$ are all independent their coefficients must be separately zero. Hence we must have
$$P_{m+1} = P_{m+2} = \dots = P_n = 0. \quad \dots(8)$$

Thus we get $m + n$ equations
$$f_1 = f_2 = f_3 = \dots = f_m = 0 \text{ and } P_1 = P_2 = \dots = P_n = 0$$
which determine the m multipliers $\lambda_1, \lambda_2, \dots, \lambda_m$ and values of $x_1, x_2, \dots, x_n$ for which maxima or minima of u are possible.

§ 1.4-7. APPLICATION OF THE METHOD OF UNDETERMINED MULTIPLIERS

Although the method explained in § 1.4-6 can be applied to determine the extreme (maxima or minima) values of the given function, however it is more convenient to find out the extreme values of u with the help of a new function F defined by
$$F = \phi + \lambda_1 f_1 + \lambda_2 f_2 + \dots + \lambda_m f_m.$$

This method has the advantage over the method given in § 1.1-6 in the sense that it enables us to decide whether the values are maximum or minimum. Although the method is quite general, however, we give below the method for three variables x, y, z connected by two relations.

Let $u = \phi(x, y, z)$ be subjected to the following conditions :
$$f_1(x, y, z) = 0 \quad \dots(1)$$
and
$$f_2(x, y, z) = 0. \quad \dots(2)$$

We now define the following auxiliary function $F(x, y, z)$ by
$$F(x, y, z) = \phi(x, y, z) + \lambda_1 f_1(x, y, z) + \lambda_2 f_2(x, y, z)$$
where x, y, z are independent variables. Parameters λ_1 and λ_2 are independent (from x, y, z) are called Lagrange's multipliers.

Now for maxima or minima of $F(x, y, z)$, we have
$$\frac{\partial F}{\partial x} = \frac{\partial \phi}{\partial x} + \lambda_1 \frac{\partial f_1}{\partial x} + \lambda_2 \frac{\partial f_2}{\partial x} = 0 \quad \dots(3)$$
$$\frac{\partial F}{\partial y} = \frac{\partial \phi}{\partial y} + \lambda_1 \frac{\partial f_1}{\partial y} + \lambda_2 \frac{\partial f_2}{\partial y} = 0 \quad \dots(4)$$
and
$$\frac{\partial F}{\partial z} = \frac{\partial \phi}{\partial z} + \lambda_1 \frac{\partial f_1}{\partial z} + \lambda_2 \frac{\partial f_2}{\partial z} = 0. \quad \dots(5)$$

Equations (1), (2), (3), (4) and (5) are conditions for maxima or minima of $F(x, y, z)$ and solving them we get the values of $\lambda_1, \lambda_2, x, y, z$.

Again from equations (3), (4) and (5) it is clear that the maxima or minima values of the function $\phi(x, y, z)$ are the same as that of the function $F(x, y, z)$, while all variables are assumed to be independent in $\phi(x, y, z)$.

Interpretation of Maxima or Minima. For the interpretation of maxima or minima we adopt the following procedure :
$$d^2u = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z}\right)^2 \phi + \frac{\partial \phi}{\partial x} d^2x + \frac{\partial \phi}{\partial y} d^2y + \frac{\partial \phi}{\partial z} d^2z \quad \dots(6)$$

$$d^2f_1=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}+dz\frac{\partial}{\partial z}\right)^2 f_1+\frac{\partial f_1}{\partial x}d^2x+\frac{\partial f_1}{\partial y}d^2y+\frac{\partial f_1}{\partial z}d^2z=0 \quad ...(7)$$

$$d^2f_2=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}+dz\frac{\partial}{\partial z}\right)^2 f_2+\frac{\partial f_2}{\partial x}d^2x+\frac{\partial f_2}{\partial y}d^2y+\frac{\partial f_2}{\partial z}d^2z=0. \quad ...(8)$$

…ltiplying equation (7) by λ_1 and equation (8) by λ_2 and their sum is added to eqution (6), we get

$$d^2u=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}+dz\frac{\partial}{\partial z}\right)^2(\phi+\lambda_1 f_1+\lambda_2 f_2)=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}+dz\frac{\partial}{\partial z}\right)^2 F$$

$$d^2u=d^2F. \quad ...(9)$$

…us d^2u and d^2F are equal, where d^2F is obtained by assuming all the variables x, y, z as independent. …equation (9) it is clear that the signs of d^2u and d^2F are same. Hence the value of $u=\phi(x,y,z)$ will …ximum or minimum according as $F(x, y, z)$ is maximum or minimum.

ILLUSTRATIVE EXAMPLES

Ex. 1. (a) *Find the maxima and minima of* $u=x^2+y^2+z^2$ *where* $ax^2+by^2+cz^2=1$.

Sol. Here
$$u=x^2+y^2+z^2, \quad ...(1)$$
$$ax^2+by^2+cz^2=1. \quad ...(2)$$

For maxima or minima of u, we have
$$du=2x\,dx+2y\,dy+2z\,dz=0$$
$$x\,dx+y\,dy+z\,dz=0. \quad ...(3)$$

Also (2) gives,
$$ax\,dx+by\,dy+cz\,dz=0. \quad ...(4)$$

Multiplying (4) by λ and adding to (3) and then equating the coefficients of dx, dy, dz to zero, we get
$$x+a\lambda x=0 \quad ...(5)$$
$$y+b\lambda y=0 \quad ...(6)$$
$$z+c\lambda z=0. \quad ...(7)$$

Multiplying (5), (6) and (7) by x, y and z respectively and then adding, we obtain
$$(x^2+y^2+z^2)+\lambda(ax^2+by^2+cz^2)=0$$
$$\Rightarrow \quad u+\lambda=0 \Rightarrow \lambda=-u.$$

Substituting this value of λ in (5), (6) and (7), we get
$$1-au=0,\ 1-bu=0,\ 1-cu=0$$
$$\Rightarrow \quad \frac{1}{a}-u=0,\ \frac{1}{b}-u=0,\ \frac{1}{c}-u=0.$$

Hence the maximum and minimum values of u are the roots of the equation
$$\left(\frac{1}{a}-u\right)\left(\frac{1}{b}-u\right)\left(\frac{1}{c}-u\right)=0.$$

Ex. 1. (b) *Find the minimum value of* $x^2+y^2+z^2$ *having given* $ax+by+cz=p$.

Sol. Let
$$u=x^2+y^2+z^2 \quad ...(1)$$
$$ax+by+cz=p. \quad ...(2)$$

For maxima or minima of u, we have
$$du=2x\,dx+2y\,dy+2z\,dz=0$$
$$x\,dx+y\,dy+z\,dz=0. \quad ...(3)$$

Also from (2), we get
$$a\,dx+b\,dy+c\,dz=0. \quad ...(4)$$

Multiplying (3) and (4) by 1 and λ respectively and adding and then equating to zero the coefficients of dx, dy, dz, we get
$$x+\lambda a=0,\ y+\lambda b=0,\ z+\lambda c=0$$
$$\Rightarrow \quad -\lambda=\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$$
$$\Rightarrow \quad \frac{ax}{a^2}=\frac{by}{b^2}=\frac{cz}{c^2}=\frac{ax+by+cz}{a^2+b^2+c^2}=\frac{p}{a^2+b^2+c^2}$$
$$\Rightarrow \quad x=\frac{ap}{a^2+b^2+c^2},\ y=\frac{bp}{a^2+b^2+c^2},\ z=\frac{cp}{a^2+b^2+c^2} \quad ...(5)$$
$$\therefore \quad u=x^2+y^2+z^2=\frac{(a^2+b^2+c^2)\,p^2}{(a^2+b^2+c^2)^2}.$$

Hence the extreme value of u is $=\dfrac{p^2}{a^2+b^2+c^2}$.

Now to discuss max. or min. : Since there is one relation between x, y, z only two variables are to be regarded as independent.

Let x, y be the independent variables and z a function of x and y. Then (1) gives
$$\frac{\partial u}{\partial x}=2x+2z\cdot\left(\frac{\partial z}{\partial x}\right).$$

Also from (2), we get
$$a+c\frac{\partial z}{\partial x}=0 \Rightarrow \frac{\partial z}{\partial x}=-\frac{a}{c}.$$
$$\therefore \quad \frac{\partial u}{\partial x}=2x-\frac{2za}{c}.$$

Now
$$r=\frac{\partial^2 u}{\partial x^2}=2-\frac{2a}{c}\frac{\partial z}{\partial x}=2+\frac{2a^2}{c^2}=+\text{ve}.$$

Similarly
$$s=\frac{\partial^2 u}{\partial y\partial x}=-\frac{2a}{c}\frac{\partial z}{\partial y}=-\frac{2a}{c}\left(-\frac{b}{c}\right)=\frac{2ab}{c^2},$$

and
$$t=\frac{\partial^2 u}{\partial y^2}=2+\frac{2b^2}{c^2}.$$

$$\therefore \quad rt-s^2=\frac{4(a^2+c^2)(b^2+c^2)}{c^4}-\frac{4a^2b^2}{c^4}=\frac{4}{c^4}\cdot(a^2c^2+b^2c^2+c^4)$$
$$=\frac{4}{c^2}\cdot(a^2+b^2+c^2)=+\text{ve}$$

Hence u is minimum when x, y, z are given by (5) and $u_{min}=\dfrac{p^2}{a^2+b^2+c^2}$.

Ex. 2. (a) *If* a, b, c *are positive numbers, find the maximum value of* $f(x, y, z)=x^ay^bz^c$ *subject to the condition* $x+y+z=1$.

Sol. Let
$$u=x^ay^bz^c. \quad ...(1)$$

Taking log on both sides, we get
$$\log u=a\log x+b\log y+c\log z \quad \text{and} \quad x+y+z=1. \quad ...(2)$$

On differentiating, we get
$$\frac{1}{u}du=\frac{a}{x}dx+\frac{b}{y}dy+\frac{c}{z}dz$$
and
$$dx+dy+dz=0.$$

When $x=y=z=\frac{\pi}{3}$, then $r=-2\cdot\frac{1}{2}\cdots$

So that $rt-s^2=1-\frac{1}{4}=\frac{3}{4}=+$ ve and r is $-$ ve.

Hence u is maximum when $A=B=C=\frac{\pi}{3}$ and $u_{max}=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{8}$.

Ex. 4. *Show that the maximum and minimum of radii vectors of the section of the surface*

$$(x^2+y^2+z^2)^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$$

by the plane $$\lambda x+\mu y+\nu z=0$$

are given by the equation $$\frac{a^2\lambda^2}{1-a^2r^2}+\frac{b^2\mu^2}{1-b^2r^2}+\frac{c^2\nu^2}{1-c^2r^2}=0.$$

Sol. Here we have to find the maximum value of r, where

$$r^2=x^2+y^2+z^2.$$

We are given that

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=r^4$$

nd $$\lambda x+\mu y+\nu z=0.$$

Differentiating (1), (2) and (3) and using the condition for maxima or minima of r, *i.e.*, $dr=0$, we

$$x\,dx+y\,dy+z\,dz=0$$

$$\frac{x}{a^2}dx+\frac{y}{b^2}dy+\frac{z}{c^2}dz=0$$

nd $$\lambda\,dx+\mu\,dy+\nu\,dz=0.$$

Multiplying (4), (5), (6) by $1, \lambda_1, \lambda_2$ respectively and adding and then equating to zero the coeffic

dx, dy, dz, we get

Ex. 5. *Find the maximum and minimum values of* $\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}$ *when*

$$lx+my+nz=0 \text{ and } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Interpret the result geometrically.

Sol. Let $$u=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4} \quad ...(1)$$

when $$lx+my+nz=0 \text{ and } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \quad ...(2)$$

Hence for maxima or minima of u, we have

$$du=\frac{2x}{a^4}dx+\frac{2y}{b^4}dy+\frac{2z}{c^4}dz=0$$

or $$\frac{x}{a^4}dx+\frac{y}{b^4}dy+\frac{z}{c^4}dz=0. \quad ...(3)$$

Also (2) gives

$$l\,dx+m\,dy+n\,dz=0 \quad ...(4)$$

and $$\frac{x}{a^2}dx+\frac{y}{b^2}dy+\frac{z}{c^2}dz=0. \quad ...(5)$$

Multiplying (3), (4) and (5) by $1, \lambda_1, \lambda_2$ respectively and adding and then equating the coefficients of dx, dy and dz to zero, we get

$$\frac{x}{a^4}+\lambda_1 l+\lambda_2\cdot\frac{x}{a^2}=0 \quad ...(6)$$

$$\frac{y}{b^4}+\lambda_1 m+\lambda_2\cdot\frac{y}{b^2}=0 \quad ...(7)$$

and $$\frac{z}{c^4}+\lambda_1 n+\lambda_2\cdot\frac{z}{c^2}=0. \quad ...(8)$$

Chapter 5

Curvature, Radius of Curvature, Centre of Curvature

§ 1.5-1. SOME DEFINITIONS

(A) **Curvature.** Let P and Q be two neighbouring points on a given continuous curve such that arc $PQ = \delta s$ and arc $AP = s$, where A is some fixed point on the curve.

Again suppose that the tangents at the points P and Q make angles ψ and $\psi + \delta\psi$ respectively with the axis of x. Then

(i) $\delta\psi$ is called the **total curvature** of the arc PQ,

(ii) the ratio $\delta\psi/\delta s$ is called the **average curvature** of the arc PQ.

(iii) $\lim_{\delta s \to 0} \frac{\delta\psi}{\delta s}$ *i.e.*, $\frac{d\psi}{ds}$ is called the **curvature** of the curve at the point P and is usually denoted by the Greek letter κ (read as Kappa).

(iv) the reciprocal of **curvature** is called the **radius of** curvature and is usually denoted by the Greek letter ρ.

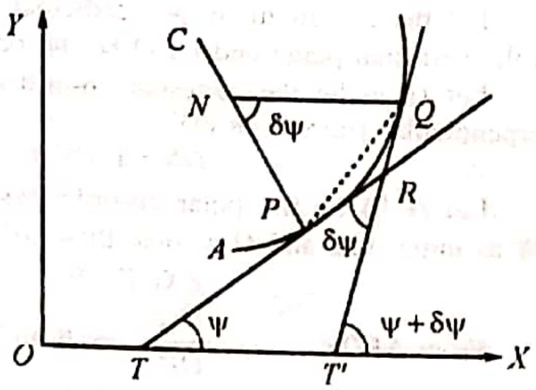

Fig. 1.5

Radius of curvature. Suppose P is a given point on a given curve. Take an other point Q on it in the neighbourhood of P. Suppose that the normals at P and Q intersect at the point N. If N approaches to a definite position C when $Q \to P$, then C is called the **Centre of Curvature** of the curve at the point P.

The distance CP is defined as the **radius** of curvature of the curve at the point P.

Note. The point Q may tend to P from the right or from the left (above), it makes no difference.

Circle of Curvature. A circle whose centre is the centre of curvature C and radius is radius of curvature CP, is called the circle of curvature of the curve at the point P.

Every chord of this circle passing through P is called the **chord of curvature.**

Intrinsic equation. The relation between s and ψ for any curve is called the **intrinsic equation** of the curve.

A formula expressed in terms of s and ψ is called the **intrinsic formula.**

Polar Co-ordinates :

In order to specify the position of a point in a plane, in polar co-ordiantes, we take a fixed half straight line OX in the plane and a fixed point O on this straight line.

The fixed line OX is called the **initial line** and the fixed point O is called the **pole.**

Now join the point O to P. The distance $OP = r$ is called the **radius vector** and $\angle XOP = \theta$ the **vectorial angle** of the point P, then (r, θ) are called the **polar co-ordinates** of the point P.

For any point $P(r, \theta)$ the vectorial angle θ is taken to be **positive** when measured in the anti-clockwise direction from OX and θ is taken to be **negative** when measured in the clockwise direction from OX. The radius vector r is taken to be **positive** when measured away from the pole O along the bounding line of the vectorial angle and r is taken to be negative if measured backwards along this line.

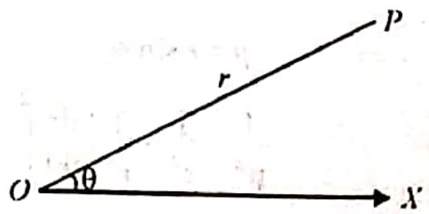

Fig. 1.6

ordered pair (r, θ) of real numbers is given then it corresponds to one and only one point.
ne converse is not true *i.e.*, if a point P whose polar co-ordinates
s given then its polar co-ordinates are also
$(r, \theta + 2\pi)$; $(r, \theta + 4\pi)$ ..., $(-r, \theta + \pi)$, $(-r, \theta + 3\pi)$, etc.
if we produce OP backwards to a point P' on it such that
, then the polar co-ordinates of P' are $(-r, \theta)$.
polar co-ordinates of P' may also be taken as $(r, \theta + \pi)$;
$2\pi)$; $[r, -(\pi - \theta)]$, ... etc.
general, the point $P(r, \theta)$ may be represented as follows :

$$(r, \theta + 2n\pi) \text{ or } (-r, \theta + (2n+1)\pi).$$

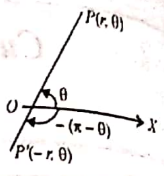

Fig. 1.7

n Between Cartesian and Polar Co-ordinates :

the two mutually perpendicular lines $X'OX$ and $Y'OY$ meet at O
artesian plane and let O be the origin.
(x, y) be the cartesian co-ordinates of any point P. Draw PN
icular from P on OX.

$$ON = x, PN = y.$$

et (r, θ) be the polar co-ordinates of the point P with respect to
initial line and O as pole then $OP = r$,

$$\angle XOP = \theta.$$

rom ΔPON, $\quad \dfrac{NO}{OP} = \cos\theta$ and $\dfrac{PN}{OP} = \sin\theta$

$$x/r = \cos\theta, \; y/r = \sin\theta$$

$$x = r\cos\theta, \quad y = r\sin\theta. \qquad ...(1)$$

Also $\qquad r^2 = x^2 + y^2, \; \tan\theta = y/x. \qquad ...(2)$

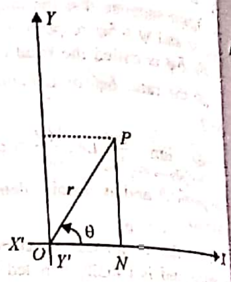

Fig. 1.8

Now with the help of the relations (1) above a cartesian equation can be transformed to polar equation
with the help of relations (2) above, a polar equation can be transformed to cartesian equation.

Note. A point P with polar co-ordinates (r, θ) is usually written as $P(r, \theta)$.

le Between Tangent and Radius Vector :

It is denoted by ϕ and is given by

$$\Rightarrow \quad \tan\phi = \frac{r d\theta}{dr} \quad \text{or} \quad \cot\phi = \frac{dr}{r d\theta}.$$

Note. From the figure it is clear that $\psi = \theta + \phi$.

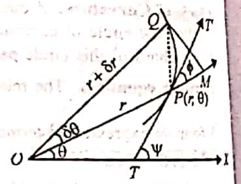

Fig. 1.9

ngth Of Perpendicular From The Pole To The Tangent :

It is denoted by p and is given by

$$\Rightarrow \quad p = r\sin\phi$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

or $\quad 1/p^2 = u^2 + (du/d\theta)^2$ where $u = 1/r$

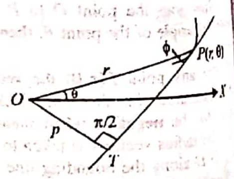

Fig. 1.10

Pedal Equation :

Definition. *For a given curve, the relation between p and r is called the pedal equation of that curve, where p is the length of the perpendicular from the pole to the tangent at any point P(r, θ) of the curve.*

§ 1.5-2. INTRINSIC FORMULA FOR THE RADIUS OF CURVATURE

See figure 1.5 of § 1.5-1 above.

Let P and Q be two neighbouring points on the given curve such that the arc $PQ = \delta s$ and arc $AP = s$, where A is some fixed point on the curve.

Let the tangents at P and Q make angles ψ and $\psi + \delta\psi$ respectively with the x-axis. Now if the tangents at P and Q intersect at R and normals at these points intersect at N, then

$$\angle PNQ = \angle T'RT = \delta\psi.$$

Let the limiting position of N be C when $Q \to P$. Therefore the radius of curvature of the curve at $P = PC = \rho = \lim\limits_{Q \to P} PN.$

Now by sine-formula in ΔPNQ, we get

$$(PN/\sin\angle NQP = \text{chord } PQ/\sin\delta\psi)$$

$$\Rightarrow \quad PN = \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \frac{\delta s}{\delta\psi} \cdot \frac{\delta\psi}{\sin\delta\psi} \cdot \sin\angle NQP \qquad [\because \text{ arc } PQ = \delta s]$$

$$\Rightarrow \quad \rho = \lim_{Q \to P} PN = \left(\lim_{Q \to P} \frac{\text{chord } PQ}{\text{arc } PQ}\right)\left(\lim_{\delta\psi \to 0} \frac{\delta s}{\delta\psi}\right)\left(\lim_{\delta\psi \to 0} \frac{\delta\psi}{\sin\delta\psi}\right)\left(\lim_{Q \to P} \sin\angle NQP\right)$$

$= 1 \, . \, (ds/d\psi) \, . \, (1) \, . \, \sin 90° = ds/d\psi$ [Since when $Q \to P$ then $\delta\psi \to 0$, and $QN \to$ Normal at P $\therefore \quad \angle NQP \to 90°$]

or $\qquad \rho = ds/d\psi.$

§ 1.5-3. CARTESIAN FORMULA FOR RADIUS OF CURVATURE

Let $y = f(x)$ be the equation of the given continuous curve.

We know that $\quad dy/dx = \tan\psi.$

Differentiating w.r.t. 'x', we have

$$\frac{d^2y}{dx^2} = \sec^2\psi \, . \, \frac{d\psi}{dx} = \sec^2\psi \, . \, \frac{d\psi}{ds} \, . \, \frac{ds}{dx} = (1 + \tan^2\psi) \, . \, \frac{1}{\rho} \, . \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{1/2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\} \, . \, \frac{1}{\rho} \, . \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{1/2}$$

$$\Rightarrow \quad \rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{d^2y/dx^2}$$

Note 1. Usually the numerical value of ρ is taken.

Note 2. From the definition of ρ it is known that its value depends only on the curve and not on the co-ordinate axes. Therefore, interchanging x and y, the following formula for ρ is obtained.

$$\rho = \frac{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{d^2x/dy^2}.$$

The above formula is useful when the tangent is parallel to y-axis i.e., $dy/dx = \infty$, or in the case x is expressed in terms of y.

Note 3. Parametric formula for radius curvature.

Let the curve be defined by

$$x = f(t) \quad \text{and} \quad y = \phi(t), \quad \text{where } t \text{ is the parameter.}$$

Then
$$\rho = \frac{[(x')^2 + (y')^2]^{3/2}}{x'y'' - y'x''}$$

where
$$x' = \frac{dx}{dt},\ y' = \frac{dy}{dt},\ x'' = \frac{d^2x}{dt^2},\ y'' = \frac{d^2y}{dt^2}$$

ILLUSTRATIVE EXAMPLES

Ex. 1. *Find the radius of curvature and curvature at any point (s, ψ) of the following curves:*

(i) $s = 4a \sin \psi$. (ii) $s = c \log \sec \psi$.

(iii) $s = a \log \cot\left(\frac{1}{4}\pi - \frac{1}{2}\psi\right) + a \sin \psi \sec^2 \psi$. (iv) $s = 8a \sin^2 (1/6)\psi$.

Sol. (i)
$$s = 4a \sin \psi.$$

∴
$$\rho = (ds/d\psi) = 4a \cos \psi \quad \text{and} \quad \kappa = \frac{1}{\rho} = \frac{1}{4a} \sec \psi.$$

(ii)
$$s = c \log \sec \psi.$$
$$\rho = \frac{ds}{d\psi} = c \cdot \frac{1}{\sec \psi} \sec \psi \tan \psi = c \tan \psi.$$

So
$$\kappa = 1/\rho = (1/c) \cot \psi.$$

(iii)
$$s = a \log \cot\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\psi\right) + a \sin \psi \sec^2 \psi.$$

∴
$$\rho = \frac{ds}{d\psi} = \frac{a\{-\operatorname{cosec}^2\left(\frac{1}{4}\pi - \frac{1}{2}\psi\right)\}\left(-\frac{1}{2}\right)}{\cot\left(\frac{1}{4}\pi - \frac{1}{2}\psi\right)} + a \cos \psi \sec^2 \psi + a \sin \psi \,.\, 2 \sec \psi \,.\, \sec \psi \tan \psi$$

$$= \frac{a}{2 \sin\left(\frac{1}{4}\pi - \frac{1}{2}\psi\right) \cos\left(\frac{1}{4}\pi - \frac{1}{2}\psi\right)} + \frac{a}{\cos \psi} + \frac{2a \sin^2 \psi}{\cos^3 \psi}$$

$$= \frac{a}{\sin 2\left(\frac{1}{4}\pi - \frac{1}{2}\psi\right)} + \frac{2a \sin^2 \psi}{\cos^3 \psi} + \frac{a}{\cos \psi}$$

$$= \frac{a}{\cos \psi} + \frac{a}{\cos \psi} + \frac{2a \sin^2 \psi}{\cos^3 \psi} = \frac{2a (\cos^2 \psi + \sin^2 \psi)}{\cos^3 \psi} = 2a \sec^3 \psi$$

So,
$$\kappa = \frac{\cos^3 \psi}{2a}$$

(iv)
$$s = 8a \sin^2 (1/6)\psi.$$

∴
$$\rho = \frac{ds}{d\psi} = 8a\left\{2 \sin \frac{1}{6}\psi \cos \frac{1}{6}\psi\right\} \cdot \frac{1}{6} = \frac{4a}{3} \sin \frac{1}{3}\psi.$$

So
$$\kappa = \frac{3}{4a} \operatorname{cosec} \frac{1}{3}\psi.$$

Ex. 2. *Find the radius of curvature at the point (x, y) of the following curves:*

(i) $y = \frac{1}{2} a(e^{x/a} + e^{-x/a}) = a \cosh (x/a)$. (ii) $y^2 = 4ax$.

(iii) $a^2 y = x^3 - a^3$. (iv) $xy = c^2$.

(v) $y = c \log \sec (x/c)$. (vi) $x^n + y^n = 1$.

Sol. (i)
$$y = a \cosh (x/a).$$

∴
$$(dy/dx) = a \{\sinh (x/a)\} (1/a) = \sinh (x/a).$$
$$(d^2y/dx^2) = (1/a) \cosh (x/a)$$

∴
$$\rho = \frac{\{1 + (dy/dx)^2\}^{3/2}}{d^2y/dx^2} = \frac{\{1 + \sinh^2 (x/a)\}^{3/2}}{(1/a) \cosh (x/a)}$$

$$= \frac{a \{\cosh^2 (x/a)\}^{3/2}}{\cosh (x/a)} = a \cosh^2 \left(\frac{x}{a}\right) = a \left(\frac{y}{a}\right)^2 = \frac{y^2}{a}$$

Again curvature
$$\kappa = \frac{1}{\rho} = \frac{a}{y^2}$$

(ii) The equation of given curve is
$$y^2 = 4ax$$

∴
$$2y \frac{dy}{dx} = 4a$$

⇒
$$\frac{dy}{dx} = \frac{2a}{y} = \frac{2a}{2\sqrt{(ax)}} = \sqrt{a}\, x^{-1/2}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2} \sqrt{a}\, x^{-3/2} = -\frac{a}{2x^{3/2}}$$

∴
$$\rho = \frac{\{1 + (dy/dx)^2\}^{3/2}}{d^2y/dx^2} = \frac{(1 + a/x)^{3/2}}{a/(2x^{3/2})} = \frac{2(x + a)^{3/2}}{\sqrt{a}}$$ [neglecting − sign]

Now curvature
$$\kappa = \frac{1}{\rho} = \frac{\sqrt{a}}{2(x + a)^{3/2}}$$

(iii)
$$a^2 y = x^3 - a^3.$$

∴
$$dy/dx = 3x^2/a^2 \text{ and } d^2y/dx^2 = 6x/a^2.$$

∴
$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1 + \frac{9x^4}{a^4}\right]^{3/2}}{\frac{6x}{a^2}} = \frac{\frac{(a^4 + 9x^4)^{3/2}}{a^6}}{\frac{6x}{a^2}} = \frac{(a^4 + 9x^4)^{3/2}}{(6x \,.\, a^4)}$$

(iv)
$$xy = c^2, \text{ or } y = c^2/x$$

$$\frac{dy}{dx} = \frac{-c^2}{x^2}, \quad \frac{d^2y}{dx^2} = \frac{2c^2}{x^3}$$

∴
$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2} = \frac{\{1 + (c^4/x^4)\}^{3/2}}{2c^2/x^3}$$

$$= \frac{x^3(x^4+c^4)^{3/2}}{2c^2x^6} = \frac{(x^4+x^2y^2)^{3/2}}{2c^2x^3} \qquad [\because c^2 = xy]$$

$$= \frac{(x^2+y^2)^{3/2}}{2c^2} = \frac{(r^2)^{3/2}}{2c^2} = r^3/(2c^2), \qquad [\because x^2+y^2=r^2]$$

is central radius vector.

(v) $y = c \log \sec (x/c)$

$(dy/dx) = [c/\sec (x/c)] \,.\, \sec (x/c) \,.\, \tan (x/c)\,(1/c) = \tan (x/c)$

$d^2y/dx^2 = (1/c) \sec^2 (x/c)$.

$$\rho = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{[1+\tan^2 (x/c)]^{3/2}}{(1/c)\sec^2 (x/c)} = \frac{c \sec^3 (x/c)}{\sec^2 (x/c)} = c \sec (x/c).$$

(vi) $x^m + y^m = 1.$ ...(1)

Differentiating (1) w.r.t. 'x', we get

$mx^{m-1} + my^{m-1}(dy/dx) = 0$

$\Rightarrow$ $dy/dx = -x^{m-1}/y^{m-1}.$ ...(2)

Again differentiating (2) w.r.t. 'x', we get

$$\frac{d^2y}{dx^2} = \frac{-(m-1)\,x^{m-2}\,y^{m-1} + (m-1)\,y^{m-2}\,(dy/dx)\,.\,x^{m-1}}{y^{2m-2}}$$

$$= \frac{(m-1)}{y^{2m-2}}\left[-x^{m-2}y^{m-1} - y^{m-2}\frac{x^{m-1}}{y^{m-1}}\,.\,x^{m-1}\right]$$

$$= \frac{-(m-1)}{y^{2m-2}}\left[x^{m-2}y^{m-1} + \frac{x^{2m-2}}{y}\right] = \frac{-(m-1)}{y^{2m-1}}[x^{m-2}y^m + x^{2m-2}]$$

$$= -(m-1)\,[x^{m-2}(1-x^m) + x^{2m-2}]/y^{2m-1} \qquad \text{[from (1)]}$$

$$= -(m-1)\,x^{m-2}/y^{2m-1}$$

$$\therefore \quad \rho = \frac{[1+x^{2m-2}/y^{2m-2}]^{3/2}}{-(m-1)\,x^{m-2}/y^{2m-1}} = \frac{[x^{2m-2}+y^{2m-2}]^{3/2}}{(1-m)\,x^{m-2}\,y^{m-2}}.$$

Ex. 3. *Find the radius of curvature of $y = 4 \sin x - \sin 2x$ at $x = \frac{1}{2}\pi$.*

Sol. Here $y = 4 \sin x - \sin 2x.$

$\therefore$ $dy/dx = 4 \cos x - 2 \cos 2x,$

$d^2y/dx^2 = -4 \sin x + 4 \sin 2x.$

$\therefore$ At $x = \frac{1}{2}\pi$, $dy/dx = 4 \cos \frac{1}{2}\pi - 2 \cos \pi = 2$

and $d^2y/dx^2 = -4 \sin \frac{1}{2}\pi + 4 \sin \pi = -4.$

$\therefore$ At $x = \frac{\pi}{2}$, $\rho = \frac{(1+4)^{3/2}}{-4} = \frac{5\sqrt{5}}{4}$ (neglecting –ve sign) **Ans.**

Ex. 4. *Find the radius of curvature of the following curves at the points indicated against them :*

(*i*) $\sqrt{x}+\sqrt{y}=\sqrt{a}$, (x, y). (*ii*) $\sqrt{x}+\sqrt{y}=\sqrt{a}$, $\left(\frac{1}{4}, \frac{1}{4}\right)$. [R.G.T.U. June 2008]

(*iii*) $y = e^x$, *at the point where it cuts the axis of y.*

Sol. (i) $\sqrt{x}+\sqrt{y}=\sqrt{a}$ ...(1)

Differentiating w.r.t. 'x',

$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}(dy/dx) = 0$

$\Rightarrow$ $dy/dx = -y^{1/2}/x^{1/2}$ ...(2)

$(d^2y/dx^2) = -\frac{1}{2}y^{-1/2}(dy/dx)(1/x^{1/2}) + (\frac{1}{2}x^{-3/2})\,.\,y^{1/2}$

$= \frac{1}{2}y^{-1/2}\,.\,(y^{1/2}/x^{1/2})\,.\,(1/x^{1/2}) + (\frac{1}{2}x^{-3/2})\,.\,y^{1/2}$

[Putting the value of (dy/dx)]

$$= \frac{1}{2x} + \frac{y^{1/2}}{2x^{3/2}} = \frac{\sqrt{x}+\sqrt{y}}{2x^{3/2}} = \frac{\sqrt{a}}{2x^{3/2}} \qquad ...(3)$$

$\therefore$ At (x, y), $\rho = \frac{(1+y/x)^{3/2}}{\sqrt{a}/(2x^{3/2})} = \frac{2(x+y)^{3/2}}{\sqrt{a}}.$

(ii) Proceeding as (i) above.

From (2), at $\left(\frac{1}{4}, \frac{1}{4}\right)\frac{dy}{dx} = -\frac{(1/4)^{1/2}}{(1/4)^{1/2}} = -1.$

From (3), at $\left(\frac{1}{4}, \frac{1}{4}\right)\frac{d^2y}{dx^2} = \frac{\sqrt{a}}{2(1/4)^{3/2}} = 4\sqrt{a}.$

$\therefore$ $\rho = [(1+1)^{3/2}/(4\sqrt{a})] = [2\sqrt{2}/(4\sqrt{a})] = [1/\sqrt{2a}]$

(iii) $y = e^x.$

$\therefore$ $dy/dx = e^x,\; d^2y/dx^2 = e^x.$

The curve crosses the y-axis where $x = 0$.

$\therefore$ $y = e^0 = 1.$

$\therefore$ At $(0, 1)$ $\rho = [(1+1)^{3/2}/1] = 2\sqrt{2}.$

Ex. 5. *Show that* $\kappa = \frac{1}{\rho} = \frac{d^2y}{dx^2}\cos^3\psi.$

Sol. We know that $dy/dx = \tan \psi$.

$$\therefore \quad \frac{d^2y}{dx^2} = \sec^2\psi\,.\,\frac{d\psi}{dx} = \sec^2\psi\,.\,\frac{d\psi}{ds}\,.\,\frac{ds}{dx}$$

or $\frac{d^2y}{dx^2} = \frac{1}{\cos^2\psi}\,.\,\frac{1}{\rho}\,.\,\frac{1}{\cos\psi}$ or $\cos^3\psi\,\frac{d^2y}{dx^2} = \frac{1}{\rho}$

$\therefore$ $\kappa = \frac{1}{\rho} = \frac{d^2y}{dx^2}\cos^3\psi.$ **Proved.**

Ex. 6. *For the curve $y = ae^{x/a}$ prove that $\rho = a \sec^2\theta \operatorname{cosec}\theta$, where $\theta = \tan^{-1}(y/a)$.*

Sol. $dy/dx = e^{x/a},\; d^2y/dx^2 = (1/a)\,e^{x/a}$

or $dy/dx = y/a,\; d^2y/dx^2 = y/a^2.$

$$\therefore \quad \rho = \frac{(1+y^2/a^2)^{3/2}}{y/a^2} = \frac{(1+\tan^2\theta)^{3/2}}{(\tan\theta)/a} \qquad [\because \text{Given } y/a = \tan\theta]$$

$$= a\sec^3\theta\,.\,\cot\theta = a\sec^2\theta\operatorname{cosec}\theta.$$

Ex. 7. *Find the radius of curvature for the curve $x = f(t)$ and $y = \psi(t)$ at any point t.*

Sol. Here $\frac{dx}{dt} = f'(t) = x'$, $\frac{dy}{dt} = \psi'(t) = y'$ then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'}{x'}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y'}{x'}\right) = \frac{d}{dt}\left(\frac{y'}{x'}\right)\cdot\frac{dt}{dx}$$

$$= \frac{x'y'' - y'x''}{(x')^2}\cdot\frac{1}{x'} = \frac{x'y'' - y'x''}{(x')^3}$$

∴ $$\rho = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{(1+y'^2/x'^2)^{3/2}}{(x'y''-y'x'')/x'^3} = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}$$ **Ans.**

Note. It can be used as a formula for ρ.

Ex. 8. (a) *Find the radius of curvature at the point 't' of the following curves :* **[R.G.T.U. Dec. 2001]**

(i) cycloid $x = a(t + \sin t), y = a(1 - \cos t)$.

(ii) $x = a\cos^3 t, y = a\sin^3 t$ or $x^{2/3} + y^{2/3} = a^{2/3}$.

Also prove that radius of curvature at any point on the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ is three times the perpendicular from the origin to the tangent at that point. **[R.G.T.U. Dec. 2002]**

(iii) $x = at^2$, $y = 2at$.

(iv) $x = a\sin 2t(1 + \cos 2t)$, $y = a\cos 2t(1 - \cos 2t)$.

(v) $x = a\sin t - b\sin(at/b)$, $y = a\cos t - b\cos(at/b)$. **[R.G.T.U. June 2004]**

(vi) $x = 3a\cos t - a\cos 3t$, $y = 3a\sin t - a\sin 3t$.

Sol. (i) $x = a(t + \sin t)$, $y = a(1 - \cos t)$

∴ $dx/dt = a(1 + \cos t)$, $dy/dt = a\sin t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\sin t}{a(1+\cos t)} = \tan\frac{1}{2}t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\tan\frac{1}{2}t\right) = \frac{1}{2}\sec^2\frac{1}{2}t\cdot\frac{dt}{dx} = \frac{\frac{1}{2}\sec^2\frac{1}{2}t}{a(1+\cos t)} = \frac{1}{2}\cdot\frac{\sec^2\frac{1}{2}t}{2a\cos^2\frac{1}{2}t} = \frac{1}{4a}\sec^4\frac{1}{2}t.$$

∴ $$\rho = \frac{\{1+(dy/dx)^2\}^{3/2}}{d^2y/dx^2} = \frac{(1+\tan^2\frac{1}{2}t)^{3/2}}{(1/4a)\sec^4\frac{1}{2}t} = 4a\cos\frac{1}{2}t.$$

Note. We have $t = 0$, at the vertex of the cycloid.

∴ $\rho = 4a\cos 0 = 4a$ at the vertex of the cycloid.

(ii) The equation of curve is

$$x^{2/3} + y^{2/3} = a^{2/3}. \quad ...(1)$$

The parametric equations of the curve (1) is

$$x = a\cos^3 t, \; y = a\sin^3 t \quad ...(2)$$

∴ $dx/dt = -3a\cos^2 t\sin t$,

$dy/dt = 3a\sin^2 t\cos t$.

∴ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\tan t$$

$$\frac{d^2y}{dx^2} = -\sec^2 t\cdot\frac{dt}{dx} = \frac{-\sec^2 t}{-3a\cos^2 t\sin t} = \frac{\sec^4 t\,\text{cosec}\, t}{3a}$$

∴ At the point 't'

$$\rho = \frac{(1+\tan^2 t)^{3/2}}{(1/3a)\sec^4 t\,\text{cosec}\, t} = \frac{3a\sec^3 t}{\sec^4 t\,\text{cosec}\, t} = 3a\sin t\cos t \quad ...(3)$$

Ans. for (2)

$= 3a\,(y/a)^{1/3}\,(x/a)^{1/3}$ [from (2)]

$= 3a^{1/3}\,x^{1/3}\,y^{1/3}$. **Ans.** for (1)

The equation of the tangent to (2) at $(a\cos^3 t, a\sin^3 t)$ is

$$y - a\sin^3 t = -\frac{\sin t}{\cos t}(x - a\cos^3 t)$$

or $$x\sin t + y\cos t = a\sin t\cos t.$$

If p is the length of perpendicular from (0, 0) to this tangent, then

$$p = \frac{a\sin t\cos t}{\sqrt{(\sin^2 t + \cos^2 t)}} = a\sin t\cos t \quad ...(4)$$

Hence from (3) and (4), $\rho = 3p$.

(iii) $x = at^2$, $y = 2at$

$dx/dt = 2at$, $dy/dt = 2a$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t},$$

∴ $$\frac{d^2y}{dx^2} = -\frac{1}{t^2}\cdot\frac{dt}{dx} \quad \text{or} \quad d^2y/dx^2 = -1/(2at^3).$$

∴ $$\rho = \frac{[1+(1/t^2)]^{3/2}}{-1/(2at^3)} = 2a\,(t^2+1)^{3/2}.$$ [neglecting –ve sign]

(iv) $x = a\sin 2t + a\sin 2t\cos 2t = a\sin 2t + \frac{1}{2}a\sin 4t$

$dx/dt = 2a\cos 2t + 2a\cos 4t = 2a\,(2\cos 3t\cos t)$

and $y = a\cos 2t - a\cos^2 2t$

$dy/dt = -2a\sin 2t + 4a\cos 2t\sin 2t = 2a\,(\sin 4t - \sin 2t) = 4a\cos 3t\sin t$

∴ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \tan t$$

$$\frac{d^2y}{dx^2} = \sec^2 t\cdot\frac{dt}{dx} = \frac{\sec^2 t}{4a\cos 3t\cos t} = \frac{\sec^3 t}{4a\cos 3t}.$$

∴ $$\rho = \frac{(1+\tan^2 t)^{3/2}}{\sec^3 t/(4a\cos 3t)} = \frac{4a\cos 3t\,.\,\sec^3 t}{\sec^3 t} = 4a\cos 3t.$$

(v) $x = a\sin t - b\sin(at/b)$, $y = a\cos t - b\cos(at/b)$

$$\frac{dx}{dt} = a\cos t - b\cos\left(\frac{at}{b}\right)\left(\frac{a}{b}\right) = a\cos t - a\cos\left(\frac{at}{b}\right)$$

$$= a\left[2\sin\frac{1}{2}t\left(\frac{a}{b}+1\right).\sin\frac{1}{2}t\left(\frac{a}{b}-1\right)\right]$$

$$\frac{dy}{dt} = -a\sin t + b\sin\left(\frac{at}{b}\right)\left(\frac{a}{b}\right) = -a\sin t + a\sin\left(\frac{at}{b}\right)$$

$$= a\left[2\cos\left\{\frac{1}{2}t\left(\frac{a}{b}+1\right)\right\}.\sin\left\{\frac{1}{2}t\left(\frac{a}{b}-1\right)\right\}\right].$$

$\therefore$ (a) Curvature κ at $(x, y) = \dfrac{d^2y/dx^2}{[1+(dy/dx)^2]^{3/2}} = \dfrac{\sqrt{\left(\dfrac{p}{2}\right)} \cdot \dfrac{1}{2x^{3/2}}}{\{1+p/(2x)\}^{3/2}} = \dfrac{\sqrt{p}}{(2x+p)^{3/2}}$

$$= \frac{p^2}{(2px+p^2)^{3/2}}. \quad ...(1)$$

(b) Putting $x = p/2$, $y = p$ in (1), the curvature at $\left(\dfrac{p}{2}, p\right) = \dfrac{p^2}{(p^2+p^2)^{3/2}} = \dfrac{1}{2\sqrt{2}p}$.

(c) Putting $x = 0$, $y = 0$ in (1), the curvature at (0, 0) is $= \dfrac{p^2}{(0+p^2)^{3/2}} = \dfrac{1}{p}$.

Ex. 15. *Show that the curvature at any point on the curve* $x = a\left(\cos\theta + \log\tan\dfrac{\theta}{2}\right)$, $y = a\sin\theta$ *is* *...tly proportional to the length of the normal intercepted between the curve and the axis of x.*

Sol. Here $x = a\left(\cos\theta + \log\tan\dfrac{\theta}{2}\right)$, $y = a\sin\theta$

$$\frac{dx}{d\theta} = a\left(-\sin\theta + \frac{1}{\tan\dfrac{\theta}{2}} \cdot \frac{1}{2}\sec^2\frac{\theta}{2}\right) = \frac{a\cos^2\theta}{\sin\theta}$$

$$\frac{dy}{d\theta} = a\cos\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \tan\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan\theta) = \sec^2\theta\,\frac{d\theta}{dx} = \frac{1}{\cos^2\theta} \times \frac{\sin\theta}{a\cos^2\theta}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{\sin\theta}{a\cos^4\theta}.$$

$$\therefore \quad \rho = \frac{(1+\tan^2\theta)^{3/2}}{\dfrac{\sin\theta}{a\cos^4\theta}} = \frac{a\cos^4\theta}{\sin\theta} \times \sec^3\theta = a\cot\theta.$$

$\therefore$ curvature $\quad \kappa = \dfrac{1}{\rho} = (1/a)\tan\theta. \quad ...(1)$

Now length of the normal l (say), intercepted between the curve and the x-axis

$$= \text{length of normal} = y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}$$

$$= a\sin\theta\sqrt{(1+(\tan^2\theta)} = a\tan\theta$$

$\Rightarrow \quad l = a\tan\theta. \quad ...(2)$

Now (1) and (2) $\Rightarrow \kappa/l = 1/a^2 \Rightarrow \kappa \propto l$.

Ex. 16. *Find the curvature at the highest point of the arch of the cycloid* $x = \theta - \sin\theta$, $y = 1 - \cos\theta$.

Sol. Here $x = \theta - \sin\theta$, $y = 1 - \cos\theta \quad ...(1)$

$dx/d\theta = 1 - \cos\theta$, $dy/d\theta = \sin\theta$.

$$dy/dx = \cot(\theta/2)$$

$\therefore$ $$\frac{d^2y}{dx^2} = -\text{cosec}^2\frac{\theta}{2}\cdot\frac{1}{2}\frac{d\theta}{dx} = -\frac{1}{4}\text{cosec}^4\left(\frac{\theta}{2}\right).$$

Now the curvature (κ) at any point 'θ' of (1)

$$= \frac{d^2y/dx^2}{\{1+(dy/dx)^2\}^{3/2}} = \frac{-\text{cosec}^4(\theta/2)}{4\{1+\cot^2(\theta/2)\}^{3/2}} = \frac{-\text{cosec}(\theta/2)}{4}$$

[Neglecting – ve sing]

$\Rightarrow$ $$\kappa = \frac{1}{4}\text{cosec}(\theta/2) \quad ...(2)$$

Now the highest point of the cycloid is that point where y is greatest.

Clearly y is greatest where $\theta = \pi$.

Therefore, putting $\theta = \pi$ in (2), the curvature (κ) at the highest point

$$= (1/4)\,\text{cosec}(\pi/2) = (1/4).$$

Ex. 17. *If x, y are functions of the arc length s, then prove that*

(i) $\rho = \dfrac{-dy/ds}{d^2x/ds^2} = \dfrac{dx/ds}{d^2y/ds^2}$ (ii) $\dfrac{1}{\rho^2} = \left(\dfrac{d^2x}{ds^2}\right)^2 + \left(\dfrac{d^2y}{ds^2}\right)^2.$

(iii) $\rho^2 = (dx/d\psi)^2 + (dy/d\psi)^2.$

Sol. (i) We know that $\cos\psi = (dx/ds).$

Differentiating w.r.t. 's', we get,

$\Rightarrow$ $$-\sin\psi\,(d\psi/ds) = d^2x/ds^2$$

$$-\sin\psi\,.(1/\rho) = (d^2x/ds^2)$$

$\Rightarrow$ $$\rho = -\frac{dy/ds}{d^2x/ds^2} \quad \left[\because \sin\psi = \frac{dy}{ds}\right] \quad ...(1)$$

Again, $\sin\psi = dy/ds.$

Differentiating w.r.t. 's', we get

$$\cos\psi\,(d\psi/ds) = d^2y/ds^2$$

$\Rightarrow$ $$\cos\psi\,.(1/\rho) = (d^2y/ds^2)$$

$\Rightarrow$ $$\rho = \frac{dx/ds}{d^2y/ds^2} \quad \left[\because \cos\psi = \frac{dx}{ds}\right] \quad ...(2)$$

(ii) Squaring (1) and (2) and adding, we have

$$(1/\rho)^2 = (d^2x/ds^2)^2 + (d^2y/ds^2)^2.$$

(iii) We know that

$$\cos\psi = \frac{dx}{ds} = \frac{dx}{d\psi}\cdot\frac{d\psi}{ds} = \frac{1}{\rho}\frac{dx}{d\psi} \quad ...(3)$$

and $$\sin\psi = \frac{dy}{ds} = \frac{dy}{d\psi}\cdot\frac{d\psi}{ds} = \frac{1}{\rho}\frac{dy}{d\psi} \quad ...(4)$$

Squaring (3) and (4) and adding, we have

$$\rho^2 = \left(\frac{dx}{d\psi}\right)^2 + \left(\frac{dy}{d\psi}\right)^2 \quad [\because \cos^2\psi + \sin^2\psi = 1]$$

Note. The results of Ex. 17 above can be used as formulae for ρ.

Ex. 18. *Prove that* $(1/\rho) = (d/dx)\,(dy/ds).$

Sol. $$\frac{d}{dx}\left(\frac{dy}{ds}\right) = \frac{d}{dx}(\sin\psi) = \cos\psi\,.\frac{d\psi}{ds}\cdot\frac{ds}{dx} = \cos\psi\,.\frac{1}{\rho}\cdot\frac{1}{\cos\psi} = 1/\rho.$$

Ex. 19. *For the following curves, prove that*

(i) *curve* $x = c\log\{s + \sqrt{s^2+c^2}\},\ y = \sqrt{s^2+c^2}$, *then* $\rho = y^2/c.$

(ii) *curve* $s^2 = 8ay$, *then* $\rho = 4a\sqrt{1-(y/2a)}.$

(iii) *curve* $s = ae^{x/a}$, *then* $a\rho = s(s^2-a^2)^{1/2}.$

Sol. (i) $$\frac{dx}{ds} = c\,.\frac{1+\frac{1}{2}(s^2+c^2)^{-1/2}(2s)}{s+\sqrt{s^2+c^2}} = \frac{c}{\sqrt{s^2+c^2}} \text{ and } \frac{dy}{ds} = \frac{s}{\sqrt{s^2+c^2}}.$$

$\therefore$ $$\frac{d^2y}{ds^2} = \frac{1\,.(s^2+c^2)^{1/2} - s\frac{1}{2}(s^2+c^2)^{-1/2}(2s)}{s^2+c^2} = \frac{c^2}{(s^2+c^2)^{3/2}}.$$

From formula (ii) of Ex. 17, above, we have

$$\rho = \frac{dx/ds}{d^2y/ds^2} = \frac{c}{\sqrt{s^2+c^2}}\cdot\frac{(s^2+c^2)^{3/2}}{c^2} = \frac{s^2+c^2}{c} = y^2/c.$$

(ii) $$s^2 = 8ay.$$

Differentiating w.r.t. 's', we have

$$2s = 8a\,(dy/ds) \Rightarrow s = 4a\sin\psi \quad [\because dy/ds = \sin\psi]$$

$\therefore$ $$\rho = ds/d\psi = 4a\cos\psi = 4a\,(1-\sin^2\psi)^{1/2}$$

$$= 4a\sqrt{\left[1-\frac{s^2}{16a^2}\right]} = 4a\sqrt{\left[1-\frac{8ay}{16a^2}\right]} = 4a\sqrt{\left[1-\frac{y}{2a}\right]}.$$

(iii) $$s = ae^{x/a}.$$

$\therefore$ $$ds/dx = e^{x/a}\,.(1/a) = e^{x/a} = s/a$$

$\Rightarrow$ $$s = a\sec\psi \quad [\because ds/dx = \sec\psi]$$

$\therefore$ $$\rho = ds/d\psi = a\sec\psi\tan\psi = s\sqrt{\sec^2\psi - 1}$$

$$= s\sqrt{s^2/a^2 - 1} = (s/a)\sqrt{s^2-a^2}$$

$\Rightarrow$ $$a\rho = s\sqrt{s^2-a^2}.$$

Ex. 20. *For the curve* $s = a\log\cot\left(\frac{1}{4}\pi - \frac{1}{2}\psi\right) + a\sin\psi\sec^2\psi$

prove that $$\rho = 2a\sec^3\psi.$$

Hence prove that $d^2y/dx^2 = 1/(2a)$ *and verify that this differential equation is satisfied by the parabola* $x^2 = 4ay.$

Sol. For first part, see Ex. 1 (iii) above, we have

$$\rho = 2a\sec^3\psi \quad ...(1)$$

Again $$\rho = \frac{\{1+(dy/dx)^2\}^{3/2}}{d^2y/dx^2} = \frac{(1+\tan^2\psi)^{3/2}}{d^2y/dx^2}$$

$$2a \sec^3\psi\,(d^2y/dx^2) = \sec^3\psi$$

$\Rightarrow$ $$d^2y/dx^2 = 1/(2a).$$

or Differentiating $x^2 = 4ay$ w.r.t. 'x', we get

$$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2a}$$

$\therefore$ we see that equations (2) and (3) are equal.

Hence (2) is satisfied by $x^2 = 4ay$.

Ex. 21. *For the curve $s = a(\sec^3\psi - 1)$ prove that $\rho = 3a \tan\psi \sec^3\psi$.*

Hence prove that $3a \dfrac{dy}{dx} \cdot \dfrac{d^2y}{dx^2} = 1.$

Prove also that this differential equation is satisfied by the curve $27ay^2 = 8x^3$.

Sol. $$\rho = ds/d\psi = 3a \sec^3\psi \tan\psi.$$

Again $$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{(1 + \tan^2\psi)^{3/2}}{d^2y/dx^2}$$

$\Rightarrow$ $$3a^2 \sec^3\psi \tan\psi\,(d^2y/dx^2) = \sec^3\psi$$

$\Rightarrow$ $$3a \tan\psi\,(d^2y/dx^2) = 1$$

$\Rightarrow$ $$3a\,(dy/dx)\,.\,(d^2y/dx^2) = 1 \qquad [\because \tan\psi = dy/dx]$$

Again for the curve $27ay^2 = 8x^3$, we have

$$54ay\,.\,(dy/dx) = 24x^2 \Rightarrow dy/dx = 4x^2/(9ay)$$

$\Rightarrow$ $$\left(\frac{dy}{dx}\right)^2 = \frac{16x^4}{81a^2y^2} = \frac{16x^4\,.\,27a}{81\,.\,a^2\,.\,8x^3} = \frac{2x}{3a}.$$

Differentiating,

$\Rightarrow$ $$2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} = \frac{2}{3a} \Rightarrow 3a\frac{dy}{dx} \cdot \frac{d^2y}{dx^2} = 1.$$

Equations (2) and (3) are equal.

§ 1.5-4. PEDAL FORMULA FOR RADIUS OF CURVATURE

We know that $p = r \sin\phi$.

Differentiating w.r.t. 'r', we have

$$dp/dr = \sin\phi + r\cos\phi\,.\,(d\phi/dr) = r\frac{d\theta}{ds} + r\,.\,\frac{dr}{ds}\,.\,\frac{d\phi}{dr}$$

$$= r\left(\frac{d\theta}{ds} + \frac{d\phi}{ds}\right) = r\frac{d}{ds}(\theta + \phi) = r\frac{d\psi}{ds} \qquad [\because \psi = \theta + \phi]$$

$\Rightarrow$ $$dp/dr = r\,.\,(1/\rho)$$

$\therefore$ $$\rho = r\,(dr/dp).$$

§ 1.5-5. POLAR FORMULA FOR RADIUS OF CURVATURE

We know that $$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

Differentiating w.r.t. r, we get

$$-\frac{2}{p^3}\frac{dp}{dr} = -\frac{2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^4}\,.\,2\left(\frac{dr}{d\theta}\right)\frac{d}{dr}\left(\frac{dr}{d\theta}\right)$$

$$= -\frac{2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{2}{r^4}\frac{dr}{d\theta}\,.\,\frac{d}{d\theta}\left(\frac{dr}{d\theta}\right)\frac{d\theta}{dr}$$

$\Rightarrow$ $$-\frac{2}{p^3}\frac{dp}{dr} = -\frac{2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{2}{r^4}\frac{d^2r}{d\theta^2}$$

$\Rightarrow$ $$\frac{r^5}{p^3}\frac{dp}{dr} = r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}$$

Now $$\rho = r\frac{dr}{dp} = \frac{r(r^5/p^3)}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}} = \frac{r^6\left\{\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}.$$ [use (1)]

Hence $$\rho = \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\left(\frac{d^2r}{d\theta^2}\right)} \qquad ...(2)$$

Cor. We take $r = 1/u$, then formula (2) becomes

$$\rho = \frac{\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\}^{3/2}}{u^3\left\{u + \frac{d^2u}{d\theta^2}\right\}} \qquad ...(3)$$

Important Note. It is obvious from § 1.5-4 and § 1.5-5 that the use of pedal formula to find ρ is more convenient than polar formula. Hence if the equation of the curve is given in polar form then it is often more simple to change it first to pedal equation and then use § 1.5-4 to find ρ.

§ 1.5-6. TANGENTIAL POLAR FORM FOR RADIUS OF CURVATURE

If the equation of the curve is given in the form $p = f(\psi)$, then it is called tangential polar form equation of the curve.

We know that $$\rho = \frac{ds}{d\psi} = r\frac{dr}{dp} \text{ and } \frac{dr}{ds} = \cos\phi.$$

So $$\frac{dp}{d\psi} = \frac{dp}{dr}\,.\,\frac{dr}{ds}\,.\,\frac{ds}{d\psi} = \frac{dp}{dr}\,.\cos\phi\,.\,\frac{r\,dr}{dp} = r\cos\phi. \qquad ...(1)$$

Again $$p = r\sin\phi. \qquad ...(2)$$

Squaring (1) and (2) and adding, we get

$$p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2. \qquad ...(3)$$

Differentiating (3), w.r.t. p, we get

$$2p+2\left(\frac{dp}{d\psi}\right)\cdot\frac{d^2p}{d\psi^2}\cdot\frac{d\psi}{dp}=2r\frac{dr}{dp}$$

$$\Rightarrow \quad r\frac{dr}{dp}=p+\frac{d^2p}{d\psi^2} \Rightarrow \rho=p+\frac{d^2p}{d\psi^2}.$$

ILLUSTRATIVE EXAMPLES

Ex. 1. *Find the radius of curvature at the point* (p, r) *of the following curves :*

(i) *ellipse* $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}-\frac{r^2}{a^2b^2}$, (ii) *cardioid* $r^3=2ap^2$,

(iii) *parabola* $p^2=ar$, (iv) *lemniscate* $r^3=a^2p$,

(v) *equiangular spiral* $p=r\sin\alpha$, (vi) *hyperbola* $pr=a^2$,

(vii) $p^2=r^4/(r^2+a^2)$.

Sol. (i) Differentiating the given equation w.r.t. 'p', we have

$$-\frac{2}{p^3}=-\frac{2r}{a^2b^2}\cdot\frac{dr}{dp}$$

$$\therefore \quad \rho=r\frac{dr}{dp}=\frac{a^2b^2}{p^3}$$ **Ans.**

(ii) $$r^3=2ap^2.$$

$$\therefore \quad 3r^2\frac{dr}{dp}=4ap.$$

$$\therefore \quad \rho=r\frac{dr}{dp}=\frac{4ap}{3r}=\frac{4a}{3r}\cdot\sqrt{\left[\left(\frac{r^3}{2a}\right)\right]}=\frac{2}{3}\sqrt{2ar}.$$ **Ans.**

(iii) $$p^2=ar \Rightarrow 2p=a\frac{dr}{dp}$$

$$\therefore \quad \rho=\frac{r\,dr}{dp}=\frac{2pr}{a}=\frac{2p^3}{a^2}$$ **Ans.**

(iv) $$r^3=a^2p \Rightarrow 3r^2\frac{dr}{dp}=a^2$$

$$\therefore \quad \rho=r\cdot\frac{dr}{dp}=\frac{a^2}{3r}$$ **Ans.**

(v) $$p=r\sin\alpha$$

$$\therefore \quad 1=\frac{dr}{dp}\sin\alpha$$

$$\therefore \quad \rho=\frac{r\,dr}{dp}=\frac{r}{\sin\alpha}=r\operatorname{cosec}\alpha.$$ **Ans.**

(vi) $$pr=a^2 \Rightarrow p=a^2/r \Rightarrow 1=-\frac{a^2}{r^2}\frac{dr}{dp}$$

$$\rho=\frac{r\,dr}{dp}=\frac{r^3}{a^2}$$ (neglecting –ve sign)

(vii) $$p^2=r^4/(r^2+a^2).$$

Differentiating w.r.t. 'r', we get

$$2p\cdot\frac{dp}{dr}=\frac{4r^3(r^2+a^2)-2r\cdot r^4}{(r^2+a^2)^2}=\frac{2r^3(r^2+2a^2)}{(r^2+a^2)^2}$$

$$\therefore \quad \rho=\frac{r\,dr}{dp}=\frac{p(r^2+a^2)^2}{r^2(r^2+2a^2)}=\frac{r^2}{(r^2+a^2)^{1/2}}\cdot\frac{(r^2+a^2)^2}{r^2(r^2+2a^2)}$$

$$=(r^2+a^2)^{3/2}/(r^2+2a^2).$$ **Ans.**

Ex. 2. *For the curve* $pa^m=r^{m+1}$, *prove that the radius of curvature varies inversely as the* $(m-1)^{th}$ *power of the radius vector.*

Sol. Here $pa^m=r^{m+1}$. Differentiating w.r.t. 'p', we have

$$a^m=(m+1)\,r^m\frac{dr}{dp},$$

$$\Rightarrow \quad \frac{dr}{dp}=\frac{a^m}{(m+1)\,r^m}.$$

$$\therefore \quad \rho=\frac{r\,dr}{dp}=\frac{r\cdot a^m}{(m+1)\,r^m}=\frac{a^m}{(m+1)\,r^{m-1}} \Rightarrow \rho\propto 1/r^{m-1}.$$ **Proved.**

Ex. 3. *For any curve prove that* $\frac{r}{\rho}=\sin\phi\left(1+\frac{d\phi}{d\theta}\right)$ *where* ρ *is radius of curvature and* $\tan\phi=r\,d\theta/dr$.

Sol. We know that $\psi=\theta+\phi$.

Differentiating w.r.t. ψ, we have

$$\frac{d\psi}{ds}=\frac{d\theta}{ds}+\frac{d\phi}{ds}=\frac{d\theta}{ds}+\frac{d\phi}{d\theta}\cdot\frac{d\theta}{ds}$$

$$\Rightarrow \quad r\cdot\frac{1}{\rho}=r\frac{d\theta}{ds}\left(1+\frac{d\phi}{d\theta}\right) \qquad \left[\because \frac{1}{\rho}=\frac{d\psi}{ds}\right]$$

$$\Rightarrow \quad \frac{r}{\rho}=\sin\phi\cdot\left(1+\frac{d\phi}{d\theta}\right).$$

Ex. 4. *For any curve prove that* $\frac{d^2r}{ds^2}=\frac{\sin^2\phi}{r}-\frac{\sin\phi}{\rho}$.

Sol. We know that $dr/ds=\cos\phi$.

$$\therefore \quad \frac{d^2r}{ds^2}=-\sin\phi\frac{d\phi}{ds}=-\sin\phi\cdot\frac{d}{ds}(\psi-\theta) \qquad [\because \psi=\theta+\phi]$$

$$=-\sin\phi\left(\frac{d\psi}{ds}-\frac{d\theta}{ds}\right)=-\sin\phi\left(\frac{1}{\rho}-\frac{\sin\phi}{r}\right)=\frac{\sin^2\phi}{r}-\frac{\sin\phi}{r}.$$

Ex. 5. *If the equation to a curve be given in polar co-ordinates and if* $u=1/r$, *prove that curvature is given by* $\left(\frac{d^2u}{d\theta^2}+u\right)=\sin^3\phi$, *where* $\tan\phi=r\,d\theta/dr$.

Sol. We know that $$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

If $u=\frac{1}{r}$ then $$\frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$$

$$\therefore \quad \frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2. \qquad ...(1)$$

§ 1.5-7. RADIUS OF CURVATURE AT THE ORIGIN

If the given curve passes through the origin then ρ at origin can be obtained by putting $x = 0$, $y = 0$ in the formula of § 1.5-3. Now we are giving some other methods.

(i) Expansion Method.

From Maclaurin's theorem, we have

$$y = (y)_0 + (y_1)_0 x + \frac{(y_2)_0}{2!} x^2 + \dots \quad \dots(1)$$

Since the curve passes through (0, 0) therefore $(y)_0 = 0$.

Let $$\left(\frac{dy}{dx}\right)_{(0,0)} = (y_1)_0 = p; \quad \left(\frac{d^2y}{dx^2}\right)_{(0,0)} = (y_2)_0 = q.$$

Then (1) may be written as

$$y = px + \frac{1}{2} qx^2 + \dots. \quad \dots(2)$$

The values of p and q can be easily obtained from (2), then ρ at origin (See § 1.5-3) is given by

$$\rho = \frac{(1 + p^2)^{3/2}}{q} \quad \dots(3)$$

(ii) Newtonian Method.

Newtonian method is used when either x-axis or y-axis is tangent at the origin.

(a) If x-axis be tangent at origin.

Then $$(y_1)_0 = \left(\frac{dy}{dx}\right)_{(0,0)} = p = 0.$$

By Maclaurin's theorem, we have

$$y = 0 + 0 \, . \, x + \frac{1}{2!} qx^2 + \frac{1}{3!} (y_3)_0 \, . \, x^3 + \dots. \quad \dots(4)$$

Now multiply (4) by $2/x^2$, we have

$$\frac{2y}{x^2} = q + \frac{2}{3!} (y_3)_0 \, x + \dots.$$

Taking $\lim_{x \to 0}$ of both sides, we get

$$\lim_{x \to 0} \frac{2y}{x^2} = q. \quad \dots(5)$$

In this case, from (3), we have

$$\rho \text{ at } (0, 0) = \frac{(1 + 0)^{3/2}}{q} = \frac{1}{q}. \quad \dots(6)$$

Hence (5) and (6), give

$$\rho \text{ at } (0, 0) = \lim_{x \to 0} \frac{x^2}{2y}. \quad \dots(7)$$

(b) If y-axis be tangent at origin, then proceeding as (a) above, we have

$$\rho \text{ at } (0, 0) = \lim_{x \to 0} \frac{y^2}{2x}. \quad \dots(8)$$

Note 1. For some curves the values of p and q are determined by substituting the value of y from Maclaurin's expansion *i.e.*, $y = px + \frac{1}{2}qx^2 + \dots$

[See Ex. 5 below.]

Note 2. If the curve passes through the origin then the tangents at origin are obtained by equating to zero the lowest degree terms from the equation of the given curve.

ILLUSTRATIVE EXAMPLES

Ex. 1. *Find the radius of curvature at the origin for the curve* $y = x^3 + 5x^2 + 6x$.

Sol. The equation of the given curve is

$$y = 6x + 5x^2 + x^3. \qquad \dots(1)$$

Clearly curve (1) passes through (0, 0).

By Maclaurin's theorem

$$y = px + \frac{1}{2}qx^2 + \dots. \qquad \dots(2)$$

Comparing (1) and (2), we get

$$p = 6, \frac{1}{2}q = 5 \Rightarrow p = 6, q = 10.$$

$\therefore$ At (0, 0),
$$\rho = \frac{(1+p^2)^{3/2}}{q} = \frac{(1+36)^{3/2}}{10} = \frac{37\sqrt{37}}{10}.$$

Ex. 2. *Show that the radii of curvature of the curve* $y^2 = x^2(a+x)/(a-x)$ *at the origin are* $\pm a\sqrt{2}$.

Sol. The equation of the given may be written as

$$y = \pm x\left(\frac{a+x}{a-x}\right)^{1/2} = \pm x\left(1+\frac{x}{a}\right)^{1/2}\left(1-\frac{x}{a}\right)^{-1/2}$$

or
$$y = \pm x\left(1+\frac{x}{2a}+\dots\right)\left(1+\frac{x}{2a}+\dots\right) \qquad \text{[By Binomial theorem]}$$

$$= \pm x\left(1+\frac{x}{2a}+\frac{x}{2a}+\dots\right) = \pm\left(x+\frac{x^2}{a}+\dots\right). \qquad \dots(1)$$

Comparing equation (1) with $y = px + \frac{1}{2}qx^2 + \dots$, we get

$$p = \pm 1, \frac{1}{2}q = \pm 1/a \text{ or } q = \pm 2/a.$$

$\therefore$ At (0, 0),
$$\rho = \frac{(1+p^2)^{3/2}}{q} = \frac{(1+1)^{3/2}}{\pm 2/a} = \pm a\sqrt{2}.$$

Ex. 3. *Show that the radii of curvature at the origin on the curve* $x^3 + y^3 = 3axy$ *is each equal to* $3a/2$.   ...(1)

Sol. Here
$$x^3 + y^3 = 3axy. \qquad \dots(2)$$

Equating to zero the lowest degree terms in (2), the tangents at the origin are given by $xy = 0$ *i.e.*, $x = 0, y = 0$.

$\therefore$ By Newton's formula, we have

$$\rho_1 = \lim_{x\to 0}\frac{x^2}{2y} \text{ and } \rho_2 = \lim_{y\to 0}\frac{y^2}{2x}$$

Now dividing equation (1) by $2xy$, we get

$$\Rightarrow \lim_{x\to 0}\frac{x^2}{2y} + \lim_{x\to 0}\frac{1}{4}xy\cdot\frac{2y}{x^2} = \frac{3a}{2}$$

$$\Rightarrow \rho_1 + 0\cdot\frac{1}{\rho_1} = \frac{3a}{2} \Rightarrow \rho_1 = (3/2)a.$$

Similarly,
$$\rho_2 = (3/2)a = \lim_{x\to 0}\frac{y^2}{2x}.$$

Ex. 4. *Find the radii of curvature at the origin of the following curve :*

$$5x^3 + 7y^3 + 4x^2y + xy^2 + 2x^2 + 3xy + y^2 + 4x = 0.$$

Sol. Equating to zero the lowest degree term from the equation of curve, the equation of tangent at the origin is $x = 0$ *i.e.*, y-axis.

Now dividing the equation of curve by $2x$, we have

$$\frac{5x^2}{2} + 7y\left(\frac{y^2}{2x}\right) + 2xy + \frac{y^2}{2} + x + 3\cdot\frac{y}{2} + \frac{y^2}{2x} + 2 = 0.$$

Taking $\lim_{x\to 0}$ (also $y \to 0$) at the origin, we have

$$\lim_{x\to 0}\frac{y^2}{2x} + 2 = 0 \Rightarrow \rho = \lim_{x\to 0}\frac{y^2}{2x} = 2 \text{ (numerically).}$$

Ex. 5. *Find the radii of curvature at the origin for the curve* $y^2 - 3xy + 2x^2 - x^3 + y^4 = 0$.

Sol. Here
$$y^2 - 3xy + 2x^2 - x^3 + y^4 = 0. \qquad \dots(A)$$

Putting $y = px + \frac{1}{2}qx^2 + \dots$ in (A), we get

$$\left(px + q\frac{x^2}{2} + \dots\right)^2 - 3x\left(px + q\frac{x^2}{2} + \dots\right) + 2x^2 - x^3 + \left(px + q\frac{x^2}{2} + \dots\right)^4 = 0$$

or
$$(p^2 - 3p + 2)x^2 + \left(pq - \frac{3}{2}q - 1\right)x^3 + \dots = 0.$$

Equating to zero the coefficients of x^2 and x^3, we get

$$p^2 - 3p + 2 = 0 \qquad \dots(1)$$
$$pq - 3q/2 = 1 \qquad \dots(2)$$

Equation (1) $\Rightarrow$ $p = 1, 2.$

Putting the value of p in (2), when $p = 1$ then $q = -2$; when $p = 2$ then $q = 2$.

$\therefore$ using the formula $\rho = \frac{(1+p^2)^{3/2}}{q}$ the radii of curvature at (0, 0) are

(i) when $p = 2, q = 2$, then
$$\rho_1 = \frac{(1+4)^{3/2}}{2} = \frac{5\sqrt{5}}{2}.$$

and (ii) when $p = 1, q = -2$, then
$$\rho_2 = \frac{(1+1)^{3/2}}{-2} = -\sqrt{2} = \sqrt{2} \text{ (Numerically).}$$

Ex. 6. *Find the radii of curvature of the curve* $r = a\sin n\theta$ *at the pole.*

Sol. At the pole $r = 0$. Thus for, the curve $r = a\sin n\theta$, at the pole we have $\sin n\theta = 0$ *i.e.*, $\theta = 0$. Hence the initial line *i.e.*, x-axis is tangent at the pole. Therefore by Newton's method, we have

At the pole,
$$\rho = \lim_{x\to 0}\left(\frac{x^2}{2y}\right) = \lim_{\theta\to 0}\frac{r^2\cos^2\theta}{2r\sin\theta}$$

$$= \lim_{\theta \to 0} \frac{r \cos^2 \theta}{2 \sin \theta} = \lim_{\theta \to 0} \frac{a \sin n\theta \cos^2 \theta}{2 \sin \theta}$$

$$= \lim_{\theta \to 0} \frac{1}{2} \cdot \left(\frac{\sin n\theta}{n\theta}\right)\left(\frac{\theta}{\sin \theta}\right)(an) \cos^2 \theta = \frac{1}{2}(1)(1)(an) \cdot 1 = \frac{1}{2} an.$$

Ex. 7. *Apply Newton's formulae to find the radius of curvature at the origin of the cycloid* $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$

Sol. Here
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a \sin \theta}{a(1 + \cos \theta)} = \tan \frac{\theta}{2}.$$

Now at the origin $x = 0$, $y = 0$, therefore, from the given curve we have $\theta + \sin \theta = 0$ and 1 ... These equations give $\theta = 0$. Hence from (1), we have

$$\frac{dy}{dx} \text{ at the origin} = \tan 0 = 0.$$

This shows that the tangent at the origin is x-axis.

So ρ at the origin $= \lim_{x \to 0} (x^2/2y) = \lim_{\theta \to 0} \frac{a^2(\theta + \sin \theta)^2}{2a(1 - \cos \theta)} = \lim_{\theta \to 0} \frac{a}{2} \cdot \left[\frac{\theta^2 + 2\theta \sin \theta + \sin^2 \theta}{2 \sin^2 \frac{\theta}{2}}\right]$

$$= \lim_{\theta \to 0} \left[\left\{\frac{\theta/2}{\sin(\theta/2)}\right\}^2 + 2\left(\frac{\theta/2}{\sin \theta/2}\right) \cdot \cos \frac{\theta}{2} + \cos^2 \frac{\theta}{2}\right] = 4a.$$

Ex. 8. *Find the radius of curvature at the origin for the curve* $x = t - \frac{1}{3}t^3$, $y = t^2$.

Sol. At the origin, we have $t = 0$, from given equations.

Also $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{1 - t^2} = 0$ when $t = 0$.

This shows that the tangent at the origin is x-axis.

$$\therefore \quad \rho \text{ at } (0, 0) = \lim_{x \to 0} (x^2/2y) = \lim_{t \to 0} \frac{(t - \frac{1}{3}t^3)^2}{2t^2} = \lim_{t \to 0} \frac{1}{2}\left(1 - \frac{t^2}{3}\right)^2 = \frac{1}{2}.$$

§ 1.5-8. CENTRE OF CURVATURE

Let $y = f(x)$ be the given curve. Let $C(\alpha, \beta)$ be the centre of curvature at the point $P(x, y)$ of the ... Let $Q(x + \delta x, y + \delta y)$ be the neighbouring point of P. [See figure of § 1.5-1]. If normals at P and Q ... at N, then as $Q \to P$ then $N \to C$. Now equation of normal at $P(x, y)$ is

$$(Y - y)\,\phi(x) + (X - x) = 0$$

where $\phi(x) = dy/dx = f'(x)$.

Equation of normal at $Q(x + \delta x, y + \delta y)$ is

$$(Y - y - \delta y)\,\phi(x + \delta x) + (X - x - \delta x) = 0.$$

Subtract (1) from (2), we get

$$(Y - y)\,\{\phi(x + \delta x) - \phi(x)\} - \delta y\,\phi(x + \delta x) - \delta x = 0$$

or
$$(Y - y)\left\{\frac{\phi(x + \delta x) - \phi(x)}{\delta x}\right\} - \frac{\delta y}{\delta x}\phi(x + \delta x) - 1 = 0.$$

Now when $\delta x \to 0$, $Y \to \beta$.

Taking $\lim_{\delta x \to 0}$ of (3), we have

$$(\beta - y)\frac{d^2y}{dx^2} - \frac{dy}{dx} \cdot \phi(x) - 1 = 0$$

$$\Rightarrow \quad (\beta - y)\frac{d^2y}{dx^2} = 1 + \left(\frac{dy}{dx}\right)^2 \qquad \left[\because \phi(x) = \frac{dy}{dx}\right]$$

$$\Rightarrow \quad \beta = y + \frac{[1 + (dy/dx)^2]}{d^2y/dx^2}.$$

Again (α, β) lies on normal (1), we get

$$(\beta - y)\frac{dy}{dx} + (\alpha - x) = 0 \qquad \left[\because \phi(x) = \frac{dy}{dx}\right]$$

$$\Rightarrow \quad \alpha = x - \frac{(dy/dx)\,[1 + (dy/dx)^2]}{d^2y/dx^2}.$$

Cor. 1. The equation of circle of curvature is given by

$$(x - \alpha)^2 + (y - \beta)^2 = \rho^2.$$

Evolute. Definition. *The locus of centre of curvature* (α, β) *is called the evolute of the curve.*

§ 1.5-9. (A) CHORD OF CURVATURE THROUGH THE ORIGIN

Let $y = f(x)$ be the given curve. Let C be the centre of curvature at the point P of the curve. Then $CP = \rho$. Now draw a circle (*i.e.*, circle of curvature) with its centre at C and radius $= \rho$. Every chord through P of it, is called the chord of curvature at P.

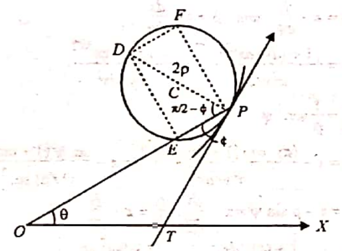

Fig. 1.14

Suppose PE is the chord of curvature through the origin O. Then

$$PE = PD \cos(\angle EPD)$$
$$= 2\rho \cos\left(\tfrac{1}{2}\pi - \phi\right)$$
$$\Rightarrow \quad \mathbf{PE = 2\rho \sin \phi}.$$

§ 1.5-9. (B) CHORD OF CURVATURE PERPENDICULAR TO RADIUS VECTOR

Let PF be the chord of curvature perpendicular to the radius vector OP.

Then $PF = DE = PD \sin\left(\tfrac{1}{2}\pi - \phi\right) = 2\rho \cos \phi$

or $\mathbf{PF = 2\rho \cos \phi}$.

§ 1.5-10. (A) CHORD OF CURVATURE PARALLEL TO X-AXIS

Clearly the chord of curvature parallel to x-axis is PM (See figure).

Then

$$PM = PD \cos(90^\circ - \psi)$$

$\therefore$ $$= 2\rho \sin \psi.$$

$\Rightarrow$ $$PM = 2\rho \sin \psi.$$

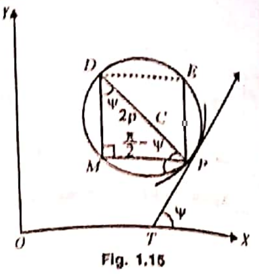

Fig. 1.16

§ 1.5-10. (B) CHORD OF CURVATURE PARALLEL TO y-AXIS

Clearly chord of curvature parallel to y-axis is PE (See fig.). Then

$$PE = DM = PD \cos \psi$$

$\Rightarrow$ $$PE = 2\rho \cos \psi.$$

ILLUSTRATIVE EXAMPLES

Ex. 1. *Prove that the co-ordinates (α, β) of the centre of curvature at any point (x, y) can be expressed in the form*

$$\alpha = x - \frac{dy}{d\psi} \text{ and } \beta = y + \frac{dx}{d\psi}.$$

Sol. We know that $$\rho = \frac{\{1 + (dy/dx)^2\}^{3/2}}{d^2y/dx^2} = \frac{(1 + \tan^2 \psi)^{3/2}}{d^2y/dx^2}.$$

$\Rightarrow$ $$\frac{d^2y}{dx^2} = \frac{1}{\rho} \sec^3 \psi.$$

Now $$\alpha = x - \frac{(dy/dx)\{1 + (dy/dx)^2\}}{d^2y/dx^2} = x - \frac{\tan \psi (1 + \tan^2 \psi)}{(1/\rho) \sec^3 \psi}$$

$$= x - \rho \sin \psi = x - \frac{ds}{d\psi} \cdot \frac{dy}{ds} = x - \frac{dy}{d\psi}$$

and $$\beta = y + \frac{1 + (dy/dx)^2}{d^2y/dx^2} = y + \frac{(1 + \tan^2 \psi)}{(1/\rho) \sec^3 \psi}$$

$$= y + \rho \cos \psi = y + \frac{ds}{d\psi} \cdot \frac{dx}{ds} = y + \frac{dx}{d\psi}.$$

Ex. 2. *Find the co-ordinates of the centre of curvature for the point (x, y) on the parabola $y^2 = 4ax$. Also find the equation of the evolute of the parabola.* **[R.G.T.U. June 2003]**

Sol. Here $$y^2 = 4ax.$$

$\Rightarrow$ $$2y \frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \sqrt{\left(\frac{a}{x}\right)}.$$

and $$\frac{d^2y}{dx^2} = -\frac{1}{2} a^{1/2} \cdot x^{-3/2}$$

$\therefore$ $$\alpha = x - \frac{\left[1 + \frac{a}{x}\right] \cdot \sqrt{\left(\frac{a}{x}\right)}}{\left(-\frac{1}{2} a^{1/2} x^{-3/2}\right)} = x + 2(x + a) = 3x + 2a \quad \text{...(1)}$$

and $$\beta = y + \frac{[1 + (a/x)]}{-\frac{1}{2} a^{1/2} x^{-3/2}} = 2a^{1/2} x^{1/2} - 2a^{-1/2} x^{1/2} (x + a)$$

$$= 2a^{-1} x^{1/2} [a - (x + a)] = -2a^{-1/2} x^{3/2}. \quad \text{...(2)}$$

Thus centre of curvature is $(3x + 2a, -2a^{-1/2} x^{3/2})$.

To find evolute. From (1)

$$x = (\alpha - 2a)/3 \quad \text{...(3)}$$

From (2), $$x^3 = a\beta^2/4 \quad \text{...(4)}$$

Putting the value of x from (3) in (4), we have

$$[(\alpha - 2a)/3]^3 = a\beta^2/4$$

or $$27 a\beta^2 = 4(\alpha - 2a)^3.$$

$\therefore$ The equation of locus of (α, β) *i.e.*, the equation of required evolute is,

$$27ay^2 = 4(x - 2a)^3.$$

Ex. 3. *Find the co-ordinates of centre of curvature at the point (x, y) of the ellipse $x^2/a^2 + y^2/b^2 = 1$.*

Sol. Here $$dy/dx = -b^2x/(a^2y).$$

$\therefore$ $$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left[\frac{y \cdot 1 - x(dy/dx)}{y^2}\right]$$

$$= \frac{-b^2}{a^2y^2}\left[y - x\left(\frac{-b^2x}{a^2y}\right)\right] = \frac{-b^2}{a^2y^2}\left(\frac{a^2y^2 + b^2x^2}{a^2y}\right)$$

$$= -\frac{b^2}{a^2y}\left(\frac{a^2b^2}{a^2y}\right) = -\frac{b^4}{a^2y^3} \quad [\because \text{ from given curve } b^2x^2 + a^2y^2 = a^2b^2]$$

$\therefore$ $$\alpha = x - \frac{\frac{-b^2x}{a^2y}\left(1 + \frac{b^4x^2}{a^4y^2}\right)}{-b^4/(a^2y^3)} = x - \frac{(b^4x^2 + a^4y^2) \cdot x}{a^4b^2}$$

and $$\beta = y + \frac{1 + (b^4x^2/a^4y^2)}{-b^4/(a^2y^3)} = y - \frac{(b^4x^2 + a^4y^2) \cdot y}{a^2b^4}.$$

Ex. 4. *Find the centre of curvature at the point $(1, -1)$ of the curve $y = x^3 - 6x^2 + 3x + 1$. Hence find the equation of the circle of cuvature at this point.*

Sol. $$dy/dx = 3x^2 - 12x + 3;$$

$$d^2y/dx^2 = 6x - 12.$$

$\therefore$ At $(1, -1)$, $dy/dx = -6$, $d^2y/dx^2 = -6$.

$\therefore$ $$\alpha = 1 - \frac{(-6)\{1 + (-6)^2\}}{(-6)} = 1 - 37 = -36$$

and $$\beta = -1 + \frac{1 + (-6)^2}{(-6)} = -\frac{43}{6} = -7\frac{1}{6}.$$

$\therefore$ Centre of curvature $\left(-36, -\frac{43}{6}\right)$.

$$\rho = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2} = -\frac{(1+6^2)^{3/2}}{-6} = \frac{37\sqrt{37}}{6}.$$

Now The equation of circle of curvature at the point (1, − 1) is

$$(x+36)^2 + \left(y+\frac{43}{6}\right)^2 = \frac{37^3}{36}$$

Ex. 5. *Find the centre of curvature at the point 't' on the ellipse $x = a\cos t,\ y = b\sin t$.*

Sol. Here $x = a\cos t,\ y = b\sin t.$

$dx/dt = -a\sin t,\ dy/dt = b\cos t$

$\therefore$ $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{b}{a}\cot t.$

$\frac{d^2y}{dx^2} = \frac{b}{a}\operatorname{cosec}^2 t \cdot \frac{dt}{dx} = -\frac{b}{a^2}\operatorname{cosec}^3 t.$

$\therefore$ Let (α, β) be the centre of curvature. Then

$$\alpha = a\cos t - \frac{\left(-\frac{b}{a}\cot t\right)\left[1+\frac{b^2}{a^2}\cot^2 t\right]}{-(b/a^2)\operatorname{cosec}^3 t}$$

$= a\cos t - (1/a)(a^2\sin^2 t + b^2\cos^2 t)\cos t$

$= (1/a)[a^2(\cos t - \sin^2 t\cos t) - b^2\cos^3 t]$

$= \frac{a^2-b^2}{a}\cos^3 t,$

and $\beta = b\sin t + \frac{1+(b^2/a^2)\cot^2 t}{(-b/a^2)\operatorname{cosec}^3 t}$

$= b\sin t - \frac{\sin t}{b}(a^2\sin^2 t + b^2\cos^2 t)$

$= \frac{b^2(\sin t - \sin t\cos^2 t) - a^2\sin^3 t}{b} = -\frac{a^2-b^2}{a}\sin^3 t.$

Ex. 6. *In the curve $y = a\log\sec(x/a)$, prove that the chord of curvature, parallel to the axis of y i*
constant length.

Sol. Here $y = a\log\sec(x/a)$

$\frac{dy}{dx} = a\cdot\frac{1}{\sec(x/a)}\cdot\sec\left(\frac{x}{a}\right)\cdot\tan\left(\frac{x}{a}\right)\cdot\frac{1}{a} = \tan\frac{x}{a}$

$\frac{d^2y}{dx^2} = \frac{1}{a}\cdot\sec^2\frac{x}{a}.$

Now chord of curvature parallel to y-axis

$$= 2\rho\cos\psi = \frac{2\rho}{\sec\psi} = \frac{2\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}\cdot\frac{1}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{1/2}}$$

$$= 2\cdot\frac{1+(dy/dx)^2}{d^2y/dx^2} = 2\cdot\frac{1+\tan^2(x/a)}{(1/a)\sec^2(x/a)} = 2a = \text{constant}.$$

Ex. 7. *Show that the chord of curvature through the pole of the curve $r = ae^{m\theta}$ is $2r$.*

Sol. Here $r = ae^{m\theta}$.

$\Rightarrow$ $\frac{dr}{d\theta} = am\cdot e^{m\theta} = mr.$

Now $\tan\phi = r\frac{d\theta}{dr} = \frac{r}{mr} = \frac{1}{m} = \tan\alpha$ (say)

$\Rightarrow$ $\phi = \alpha.$

Now $p = r\sin\phi \Rightarrow p = r\sin\alpha$ $\quad\therefore\ dp/dr = \sin\alpha.$

$\therefore$ $\rho = r\,(dr/dp) = r\operatorname{cosec}\alpha.$

$\therefore$ Required chord $= 2\rho\sin\phi = 2r\operatorname{cosec}\alpha\,.\,\sin\alpha = 2r.$

Ex. 8. *Show that the length of the chord of curvature, parallel to y-axis, at the origin in the parabola, $y = mx + (x^2/a)$ is $(1+m^2)\,a$.*

Sol. Here $y = mx + (x^2/a).$

$\therefore$ $dy/dx = m + (2x/a);\quad d^2y/dx^2 = 2/a$

at the point (0, 0) $dy/dx = m;\ d^2y/dx^2 = 2/a.$

$\therefore$ ρ at (0, 0) $= \frac{(1+m^2)^{3/2}}{2/a} = \frac{a(1+m^2)^{3/2}}{2}.$

Again at (0, 0), $dy/dx = \tan\psi \Rightarrow \tan\psi = m.$

$\therefore$ $\cos\psi = 1/\sqrt{1+\tan^2\psi} = 1/\sqrt{(1+m^2)}.$

The required chord of curvature $= 2\rho\cos\psi$

$$= 2\cdot\frac{a(1+m^2)^{3/2}}{2}\cdot\frac{1}{\sqrt{1+m^2}} = (1+m^2)\,a.$$

Ex. 9. *Show that in the curve $y = a\cosh(x/a)$, the chord of curvature parallel to the axis of x is of length $a\sinh(2x/a)$.*

Sol. Here $y = a\cosh(x/a)$

$dy/dx = \sinh(x/a),\quad d^2y/dx^2 = \frac{1}{a}\cosh\left(\frac{x}{a}\right).$

Now the length of the chord of curvature parallel to the x-axis

$$= 2\rho\sin\psi = \frac{2\rho}{\operatorname{cosec}\psi} = \frac{2\rho}{\sqrt{1+\cot^2\psi}} = \frac{2\rho\tan\psi}{\sqrt{1+\tan^2\psi}}$$

$$= 2\cdot\frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2}\cdot\frac{(dy/dx)}{\sqrt{1+(dy/dx)^2}} \qquad [\because\ dy/dx = \tan\psi]$$

$$= \frac{2\frac{dy}{dx}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{dx^2}} = \frac{2\sinh\left(\frac{x}{a}\right)\left\{1+\sinh^2\left(\frac{x}{a}\right)\right\}}{\frac{1}{a}\cosh\left(\frac{x}{a}\right)}$$

$= 2a\sinh(x/a)\cosh(x/a) = a\sinh(2x/a).$