11.5. THE STEFAN-BOLTZMANN LAW

The law states that *the emissive power of a black body is directly proprtional to the fourth power of its absolute temperature.*

i.*e.*, $E_b = \sigma T^4$ ...(11.7)

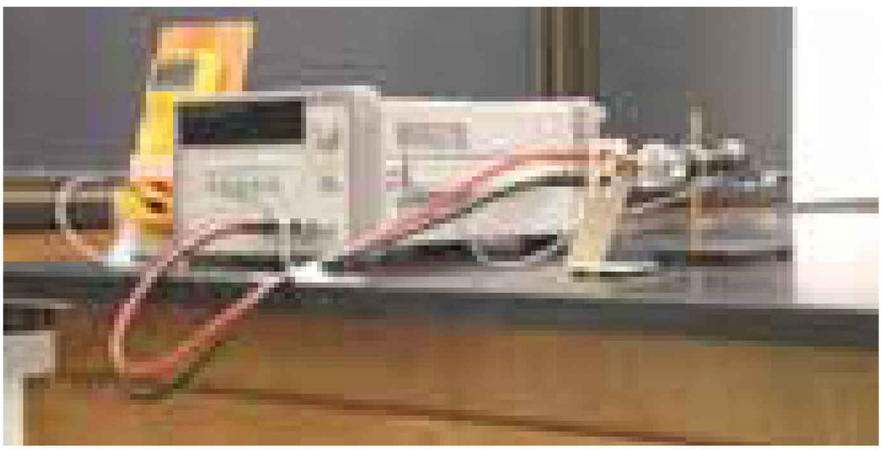
where, E_b = Emissive power of a black body, and

σ = Stefan-Boltzmann constant

$= 5.67 \times 10^{-8}$ W/m^2 K^4.

Equation (11.7) can be rewritten as:

$$E_b = 5.67 \left(\frac{T}{100}\right)^4 \qquad ...(11.8)$$



Experimental setup of Stefan-Boltzmann law

11.6. KIRCHHOFF'S LAW

The law states that *at any temperature the ratio of total emissive power E to the total absorptivity α is a constant for all substances which are in thermal equilibrium with their environment.*

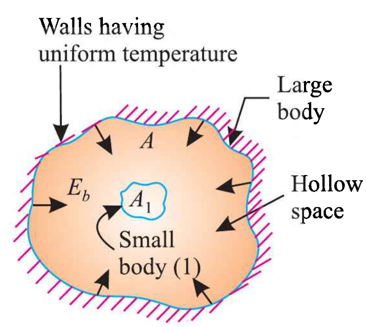


Fig. 11.5. Derivation of Kirchhoff's law.

Gustav Kirchhoff (1824-1887)

Let us consider a large radiating body of surface area A which encloses a small body (1) of surface area A_1 (as shown in Fig. 11.5). Let the energy fall on the unit surface of the body at the rate E_b. Of this energy, generally, a fraction α, will be absorbed by the small body. Thus, this energy absorbed by the small body (1) is $\alpha_1 A_1 E_b$, in which α_1 is the absorptivity of the body. When thermal equilibrium is attained, the *energy absorbed* by the body (1) must be equal to the *energy emitted*, say, E_1 per unit surface. Thus, at equilibrium, we may write

$$A_1E_1 = \alpha_1 A_1 E_b \quad \text{...(11.9)}$$

Now we remove body (1) and replace it by body (2) having absorptivity α_2. The radiative energy impinging on the surface of this body is again E_b. In this case, we may write

$$A_2E_2 = \alpha_2 A_2 E_b \quad \text{...(11.10)}$$

By considering generality of bodies, we obtain

$$E_b = \frac{E_1}{\alpha_1} = \frac{E_2}{\alpha_2} = \frac{E}{\alpha} \quad \text{...(11.11)}$$

Also, as per definition of emissivity ε, we have

$$\varepsilon = \frac{E}{E_b}$$

or,
$$E_b = \frac{E}{\varepsilon} \quad \text{...(11.12)}$$

By comparing eqns. (11.11) and (11.12), we obtain

$$\varepsilon = \alpha \quad \text{...(11.13)}$$

(α is always smaller than 1. Therefore, the emissive power E is always smaller than the emissive power of a black body at equal temperature.)

Thus, Kirchhoff's law also states that *the emissivity of a body is equal to its absorptivity when the body remains in thermal equilibrium with its surroundings*.

11.7. PLANCK'S LAW

In 1900 Max Planck showed by quantum arguments that the spectral distribution of the radiation intensity of a black body is given by

$$(E_\lambda)_b = \frac{2\pi c^2 h \lambda^{-5}}{\exp\left(\frac{ch}{\lambda k T}\right) - 1} \quad \text{... (Planck's law)} \quad \text{...(11.14)}$$

where, $(E_\lambda)_b$ = Monochromatic (single wavelength) emissive power of a black body,

c = Velocity of light in vacuum, $2.998 \times 10^8 \simeq 3 \times 10^8$ m/s

h = Planck's constant = 6.625×10^{-34} j.s

λ = Wavelength, μm

k = Boltzmann constant = 1.3805×10^{-23} J/K, and

T = Absolute temperature, K

Hence the unit of $(E_\lambda)_b$ is W/m^2. μm.

Quite often the Planck's law is written as

$$(E_\lambda)_b = \frac{C_1 \lambda^{-5}}{\exp\left(\frac{C_2}{\lambda T}\right) - 1} \quad \text{...(11.15)}$$

where, $C_1 = 2\pi c^2 h = 3.742 \times 10^8$ W.μm^4/m^2;

$$C_2 = \frac{ch}{k} = 1.4388 \times 10^4 \ \mu m K$$

Equation (11.14) is of great importance as it provides quantitative results for the radiation from a black body.

The quantity $(E_\lambda)_b$, ***monochromatic emissive power***, is defined as the *energy emitted by the black surface in all directions at a given wavelength λ per unit wavelength interval around λ*; that is, the rate of energy emission in the interval $d\lambda$ is equal to $(E_\lambda)_b \, d\lambda$. The total emissive power and monochromatic emissive power are related by the equation,

$$E_b = \int_0^\infty (E_\lambda)_b \, d\lambda \qquad ...(11.16)$$

A plot of $(E_\lambda)_b$ as a function of temperature and wavelength is given in Fig. 11.6.

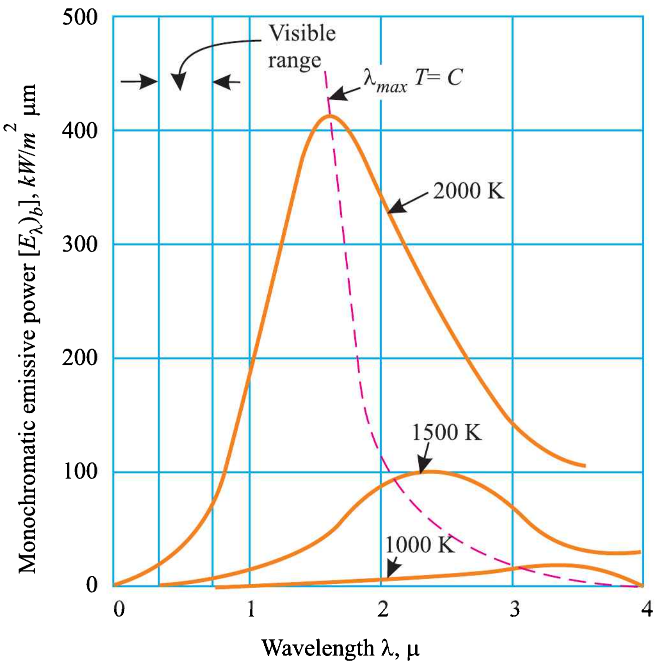


Fig. 11.6. Variation of emissive power with wavelength.

The plot shows the the following distinct characteristics of black body radiations :

1. *The energy emitted at all wavelengths increases with rise in temperature.*
2. *The peak spectral emissive power shifts towards a smaller wavelength at higher temperatures*. This shift signifies that at elevated temperature, much of the energy is emitted in a narrow band ranging on both sides of wavelength at which the monochromatic power is maximum.
3. The *area* under the monochromatic emissive power versus wavelength, at any temperature, gives *the rate of radiant energy emitted* within the wavelength interval $d\lambda$. Thus,

$$dE_b = (E_\lambda)_b \, d\lambda$$

or $$E_b = \int_{\lambda=0}^{\lambda=\infty} (E_\lambda)_b \, d\lambda \qquad \text{... over the entire range of length.}$$

This integral represents the total emissive power per unit area radiated from a black body.

11.8. WIEN'S DISPLACEMENT LAW

In 1893 Wien established a relationship between the temperature of a black body and the wavelength at which the maximum value of monochromatic emissive power occurs. A peak monochromatic emissive power occurs at a particular wavelength. **Wien's displacement law** *states that the product of* λ_{max} *and T is constant, i.e.*,

$$\lambda_{max} T = \text{constant}$$

$$(E_\lambda)_b = \frac{C_1 \lambda^{-5}}{\exp\left(\frac{C_2}{\lambda T}\right) - 1}$$

Wilhelm Wien (1864-1928)

$(E_\lambda)_b$ becomes maximum (if T remains constant) when

$$\frac{d(E_\lambda)_b}{d\lambda} = 0$$

i.e.,

$$\frac{d(E_\lambda)_b}{d\lambda} = \frac{d}{d\lambda}\left[\frac{C_1 \lambda^{-5}}{\exp\left(\frac{C_2}{\lambda T}\right) - 1}\right] = 0$$

or,

$$\frac{\left[\exp\left(\frac{C_2}{\lambda T}\right) - 1\right](-5 C_1 \lambda^{-6}) - C_1 \lambda^{-5}\left\{\exp\left(\frac{C_2}{\lambda T}\right)\frac{C_2}{T}\left(-\frac{1}{\lambda^2}\right)\right\}}{\left[\exp\left(\frac{C_2}{\lambda T}\right) - 1\right]^2} = 0$$

or,

$$-5\, C_1 \lambda^{-6} \exp\left(\frac{C_2}{\lambda T}\right) + 5 C_1 \lambda^{-6} + C_1 C_2 \lambda^{-5} \frac{1}{\lambda^2 T} \exp\left(\frac{C_2}{\lambda T}\right) = 0$$

Dividing both sides by $5C_1\, \lambda^{-6}$, we get

$$-\exp\left(\frac{C_2}{\lambda T}\right) + 1 + \frac{1}{5} C_2 \frac{1}{\lambda T} \exp\left(\frac{C_2}{\lambda T}\right) = 0$$

Solving this equation by trial and error method, we get

$$\frac{C_2}{\lambda T} = \frac{C_2}{\lambda_{max} T} = 4.965$$

$$\therefore \quad \lambda_{max} T = \frac{C_2}{4.965} = \frac{1.439 \times 10^4}{4.965}\ \mu m\, K = 2898\ \mu m\, K\ (\simeq 2900\ \mu m\, K)$$

i.e., $\lambda_{max}\, T = 2898\ \mu m K$ ...(11.18)

This law holds true for more *real substances*; there is however some deviation in the case of a metallic conductor where the product (λ_{max} T) is found to vary with absolute temperature. It is used in *predicting a very high temperature through measurement of wavelength.*

A combination of Planck's law and Wien's displacement law yields the condition for the maximum monochromatic emissive power for a blackbody.

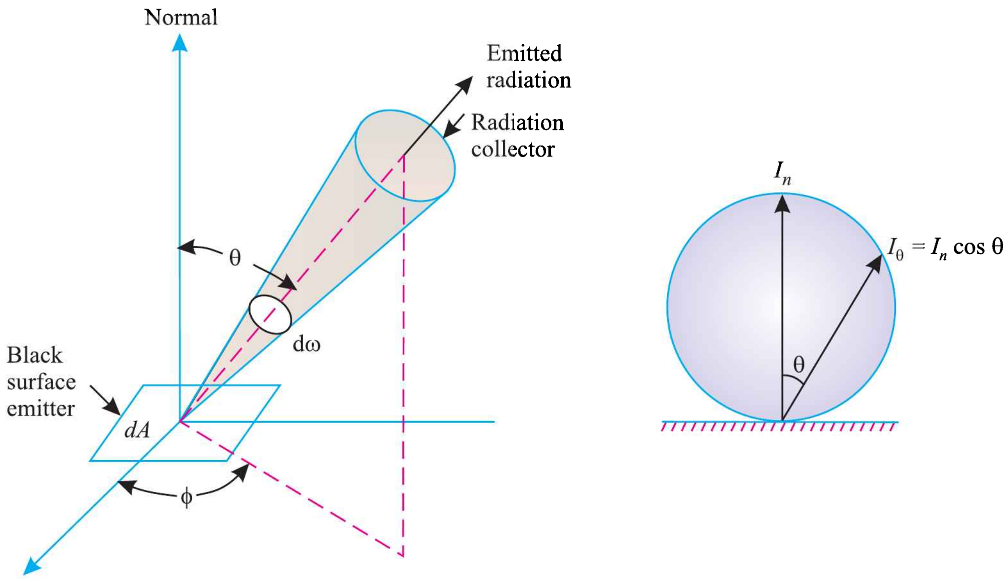
$$(E_{\lambda b})_{max} = \frac{C_1 (\lambda_{max})^{-5}}{\exp\left[\frac{C_2}{\lambda_{max} T}\right] - 1} = \frac{0.374 \times 10^{-15}\left(\frac{2.898 \times 10^{-3}}{T}\right)^{-5}}{\exp\left[\frac{1.4388 \times 10^{-2}}{2.898 \times 10^{-3}}\right] - 1}$$

or, $(E_{\lambda b})_{max} = 1.285 \times 10^{-5}\ T^5\ W/m^2$ per metre wavelength ...(11.19)

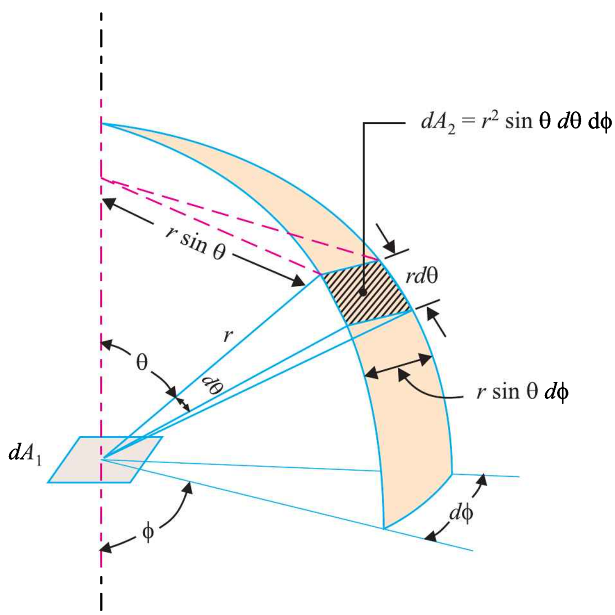
11.9. INTENSITY OF RADIATION AND LAMBERT'S COSINE LAW

11.9.1. INTENSITY OF RADIATION

When a surface element emits radiation, all of it will be intercepted by a hemispherical surface placed over the element. The **intensity of radiation** (I) is defined as the *rate of energy leaving a surface in a given direction per unit solid angle per unit area of the emitting surface normal to the mean direction in space.* A **solid angle** is defined as a *portion of the space inside a sphere enclosed by a conical surface with the vertex of the cone at the centre of the sphere.* It is measured by the ratio of the spherical surface enclosed by the cone to the square of the radius of the sphere; its unit is steradian (sr). The solid angle subtended by the complete hemisphere is given by: $\frac{2\pi r^2}{r^2} = 2\pi$.



(*a*) Spacial distribution of radiations emitted from a surface



(*b*) Illustration for evaluating area dA_2

Fig. 11.7. Radiation from an elementary surface.

Fig. 11.7 (*a*) shows a small black surface of area *dA* (emitter) emitting radiation in different directions. A black body radiation collector through which the radiation pass is located at an angular position characterised by *zenith angle* θ towards the surface normal and angle ϕ of a spherical coordinate system. Further the collector subtends a solid angle $d\omega$ when viewed from a point on the emitter.

Let us now consider radiation from the elementary area dA_1 at the centre of a sphere as shown in Fig. 11.7(*b*). Suppose this radiation is absorbed by a second elemental area dA_2, a portion of the hemispherical surface.

The projected area of dA_1 on a plane perpendicular to the line joining dA_1 and $dA_2 = dA_1 \cos\theta$.

The solid angle subtended by $dA_2 = \dfrac{dA_2}{r^2}$

$\therefore$ The intensity of radiation,

$$I = \frac{dQ_{1-2}}{dA_1 \cos\theta \times \dfrac{dA_2}{r^2}} \qquad ...(11.20)$$

where, dQ_{1-2} is the rate of radiation heat transfer from dA_1 to dA_2.

It is evident from the Fig. 11.7 (*b*) that

$$dA_2 = r d\theta\,(r \sin\theta\, d\phi)$$

or,

$$dA_2 = r^2 \sin\theta . d\theta . d\phi \qquad ...(11.21)$$

From eqns. (11.20) and (11.21), we obtain

$$dQ_{1-2} = I\, dA_1 .\sin\theta . \cos\theta .\, d\theta . d\phi$$

The total radiation through the hemisphere is given by

$$Q = I dA_1 \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\theta=0}^{\phi=2\pi} \sin\theta . \cos\theta\, d\theta\, d\phi$$

$$= 2\pi I dA_1 \int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin\theta \cos\theta\, d\theta$$

$$= \pi I dA_1 \int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin\theta \cos\theta\, d\theta$$

$$= \pi I dA_1 \int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin 2\theta\, d\theta$$

or,

$$Q = \pi I\, dA_1 \qquad ...(11.22)$$

Also

$$Q = E . dA_1$$

$\therefore$

$$E dA_1 = \pi I\, dA_1$$

or,

$$E = \pi I \qquad ...(11.23)$$

i.e., The *total emissive power of a diffuse surface is equal to* π *times its intensity of radiation.*

11.9.2. LAMBERT'S COSINE LAW

The law states that the *total emissive power* E_θ from *a radiating plane surface in any direction is directly proportional to the cosine of the angle of emission.* The angle of emission θ is the angle subtended by the normal to the radiating surface and the direction vector of emission of the receiving surface. If E_n be the total emissive power of the radiating surface in the direction of its normal, then

$$E_\theta = E_n \cos\theta \qquad ...(11.24)$$

The above equation is true only for diffuse radiation surface. The radiation emanating from a point on a surface is termed diffused if the intensity, I, is constant. This law is also known as *Lambert's law of diffuse radiation.*

Example 11.1. *The effective temperature of a body having an area of 0.12* m^2 *is 527°C. Calculate the following*:

(i) *The total rate of energy emission,*

(ii) *The intensity of normal radiation, and*

(iii) *The wavelength of maximum monochromatic emissive power.*

Solution. *Given*: $A = 0.12\ m^2$; $T = 527 + 273 = 800\ K$

(*i*) The total rate of energy emission, E_b:

$$E_b = \sigma A T^4 \text{ W (watts)} \qquad \text{...[Eqn. (11.2a)]}$$

$$= 5.67 \times 10^{-8} \times 0.12 \times (800)^4 = 5.67 \times 0.12 \times \left(\frac{800}{100}\right)^4 = \mathbf{2786.9\ W\ (Ans.)}$$

(*ii*) The intensity of normal radiation, I_{bn} :

$$I_{bn} = \frac{E_b}{\pi}, \quad \text{where } E_b \text{ is in W/m}^2 K^4 \qquad \text{...(Eqn. 11.23)}$$

$$= \frac{\sigma T^4}{\pi} = \frac{5.67 \times \left(\frac{800}{100}\right)^4}{\pi} = \mathbf{7392.5\ W/m^2 .sr\ \ (Ans.)}$$

(*iii*) The wavelength of maximum monochromatic emissive power, l_{max} :

From Wien's displacement law,

$$\lambda_{max} T = 2898\ \mu mK \qquad \text{...[Eqn. (11.18)]}$$

or,
$$\lambda_{\max} = \frac{2898}{T} = \frac{2898}{800} = \mathbf{3.622\,\mu m\ \ (Ans.)}$$

Example 11.2. *Assuming the sun to be a black body emitting radiation with maximum intensity at λ = 0.49 μm, calculate the following* :

(*i*) *The surface temperature of the sun, and*

(*ii*) *The heat flux at surface of the sun.*

Solution. *Given*: $l_{max} = 0.49\ \mu m$

(*i*) The surface temperature of the sun, *T*:

According to Wien's displacement law,

$$\lambda_{max} T = 2898\ \mu mK$$

∴
$$T = \frac{2898}{\lambda_{\max}} = \frac{2898}{0.48} = \mathbf{5914\ K\ \ (Ans.)}$$

(*ii*) The heat flux at the surface of the sun, $(E)_{sun}$:

$$(E)_{sun} = \sigma T^4 = 5.67 \times 10^{-8}\ T^4 = 5.67 \left(\frac{T}{100}\right)^4$$

$$= 5.67 \times \left(\frac{5914}{100}\right)^4 = \mathbf{6.936 \times 10^7\ W/m^2\ (Ans.)}$$

Example 11.3. *Calculate the following for an industrial furnace in the form of a black body and emitting radiation at 2500°*C :

(*i*) *Monochromatic emissive power at 1.2 μm length,*

(*ii*) *Wavelength at which the emission is maximum,*

(*iii*) *Maximum emissive power,*

(*iv*) *Total emissive power, and*

(*v*) *Total emissive power of the furnace if it is assumed as a real surface with emissivity equal to 0.9.*

Solution. *Given* : $T = 2500 + 273 = 2773 K$; $\lambda = 1.2\ \mu m$, $\varepsilon = 0.9$

(*i*) Monochromatic emissive power at 1.2 μm length, $(E_\lambda)_b$:

According to Planck's law,

$$(E_\lambda)_b = \frac{C_1 \lambda^{-5}}{\exp\left(\frac{C_2}{\lambda T}\right) - 1} \qquad \text{...[Eqn. (11.15)}$$

where, $C_1 = 3.742 \times 10^8$ W.μm^4/m^2 = 0.3742×10^{-15} W.m^4/m^2, and

$C_2 = 1.4388 \times 10^{-2}$ mK

Substituting the values, we get

$$(E_\lambda)_b = \frac{0.3742 \times 10^{-15} \times (1.2 \times 10^{-6})^{-5}}{\exp\left(\frac{1.4388 \times 10^{-2}}{1.2 \times 10^{-6} \times 2773}\right) - 1} = \frac{1.5 \times 10^{14}}{74.48} = \mathbf{2.014 \times 10^{12}\ W/m^2\ (Ans.)}$$

(*ii*) **Wavelength at which the emission is maximum, λ_{max}:**

Accodrding to Wien's displacement law,

$$\lambda_{max} = \frac{2898}{T} = \frac{2898}{2773} = \mathbf{1.045\ \mu m\ \ (Ans.)}$$

(*iii*) **Maximum emissive power, $(E_{\lambda b})$max:**

$(E_{\lambda b})_{max}$ = 1.285×10^{-5} T^5 W/m^2 per metre length [Eqn. (11.19)]

= $1.285 \times 10^{-5} \times (2773)^5$ = **2.1×10^{12} W/m^2 per metre length (Ans.)**

[**Note:** At high temperature the difference between $(E_\lambda)_b$ and $(E_{\lambda b})_{max}$ is very small].

(*iv*) **Total emissive power, E_b :**

$$E_b = \sigma T^4 = 5.67 \times 10^{-8} (2773)^4 = 5.67\left(\frac{2773}{100}\right)^4 = \mathbf{3.352 \times 10^6\ W/m^2.\ (Ans.)}$$

(*v*) **Total emissive power, E with emissivity (ε) = 0.9 :**

$$E = \varepsilon\, \sigma T^4 = 0.9 \times 5.67 \left(\frac{2773}{100}\right)^4 = \mathbf{3.017 \times 10^6\ W/m^2.\ (Ans.)}$$

Example 11.4. *Assuming the sun (diameter = 1.4 × 10^9 m) as a black body having a surface temperature of 5750 K and at a mean distance of 15 × 10^{10} m from the earth (diameter = 12.8 × 10^6 m), estimate the following:*

(i) The total energy emitted by the sun,

(ii) The emission received per m^2 just outside the atmosphere of the earth,

(iii) The total energy received by the earth if no radiation is blocked by the atmosphere of the earth, and

(iv) The energy received by a 1.6 m × 1.6 m solar collector whose normal is inclined at 50° to the sun. The energy loss through the atmosphere is 42 percent and diffuse radiation is 22 percent of direct radiation.

Solution: Radius of the sun, $r_s = \frac{1.4 \times 10^9}{2} = 0.7 \times 10^9$ m

Mean distance of the sun from the earth,

$$R = 15 \times 10^{10} \text{ m}$$

Radius of the earth $r_e = \frac{12.8 \times 10^6}{2} = 6.4 \times 10^6$ m

Surface temperature of the sun, $T = 5750\ K$

(*i*) **The total energy emitted by the sun, E_b:**

$$E_b = \sigma AT^4 = 5.67 \times 10^{-8} \times 4\pi r_s^2 \times (5750)^4$$

$$= 5.67 \times 4\pi \times (0.7 \times 10^9)^2 \times \left(\frac{5750}{100}\right)^4$$

$$= \mathbf{3.816 \times 10^{26}\ W\ (Ans.)}$$

(*ii*) The emission received per m^2:

The sun may be regarded as a point source at a distance of 15×10^{10} m from the earth. The *mean area* just outside the earth's atmosphere over which the radiation will fall is

$$= 4\pi R^2 = 4\pi \times (15 \times 10^{10})^2 \text{ m}^2$$

∴ The emission received outside the earth's atmosphere

$$= \frac{3.816 \times 10^{26}}{4\pi \times (15 \times 10^{10})^2} = \mathbf{1349.6\ W/m^2}\ \textbf{(Ans.)}$$

(*iii*) The total energy received by the earth:

Assuming the earth a spherical body, the energy received by it will be proportional to the perpendicular projected area, *i.e.*, that of a circle $(= \pi r_e^2)$.

∴ Energy received by the earth $= 1349.6 \times \pi \times (6.4 \times 10^6)^2$

$= \mathbf{1.736 \times 10^{17}}$ **W (Ans.)**

(*iv*) The energy received by the solar collector:

The direct energy reaching the earth $= (1 - 0.42) \times 1349.6 = 782.77$ W/m^2

The diffuse radiation $= 0.22 \times 782.77 = 172.21$ W/m^2

∴ Total radiation reaching the collector $= 782.77 + 172.21 \simeq 955$ W/m^2

The projected area $= A \cos\theta = 1.6 \times 1.6 \times \cos 40° = 1.961$ m^2

∴ Energy received by the solar collector

$= 955 \times 1.961 =$ **1872.7 W (Ans.)**

HIGHLIGHTS

1. '*Radiation*' *heat transfer* is defined as "the transfer of energy across a system boundary by means of an electromagnetic mechanism which is caused solely by a temperature difference.
2. The *emissive power* (E) is defined as the total amount of radiation emitted by a body per unit area per unit time; it is expressed in W/m^2.
3. *Emissivity* (ε) is defined as the ability of the surface of a body to radiate heat. It is also defined as the ratio of the emissive power of any body to the emissive power of a black body of equal temperature $\left(i.e.,\ \varepsilon = \frac{E}{E_b}\right)$.
4. A *black body* is one which neither reflects nor transmits any part of the incident radiation but absorbs all of it.
5. A *gray body* is one whose absorptivity of a surface does not vary with temperature and wavelength of the incident radiation $[\alpha = (\alpha)_\lambda = \text{constant}]$.
6. The *Stefan-Boltzmann* law states that the emissive power of a black body is directly proportional to the fourth power of its absolute temperature.

   i.e., $E_b = \sigma T^4$

   where, E_b = Emissive power of a black body, and

   σ = Stefan-Boltzmann constant $= 5.67 \times 10^{-8}$ W/m^2 K^4
7. *Kirchhoff's law* states that at any temperature the ratio of total emissive power E to the total absorptivity α is constant for all substances which are in thermal equilibrium with their environment.
8. *Planck's law* is given by:

   $$(E_\lambda)_b = \frac{C_1 \lambda^{-5}}{\exp\left[\frac{C_2}{\lambda T}\right] - 1}$$

   where, $C_1 = 2\pi c^2 h = 3.742 \times 10^8$ W.μm^4/m^2;