6.5. DIMENSIONAL ANALYSIS APPLIED TO FORCED CONVECTION HEAT TRANSFER

Let us assume that the heat transfer coefficient in a fully developed forced convection in a tube is a function of the following variables:

$$h = f(\rho, D, V, \mu, c_p, k) \quad ...(i)$$

or,

$$f_1(h, \rho, D, V, \mu, c_p, k) \quad ...(ii)$$

The physical quantities with their dimensions are as under:

S.No.	Variables	Symbols	Dimensions
1	Heat transfer coefficient	h	$MT^{-3}\theta^{-1}$
2	Fluid density	ρ	ML^{-3}
3	Tube diameter	D	L
4	Fluid velocity	V	LT^{-1}
5	Fluid viscosity	μ	$ML^{-1}T^{-1}$
6	Specific heat	c_p	$L^2T^{-2}\theta^{-1}$
7	Thermal conductivity	k	$MLT^{-3}\theta^{-1}$

Total number of variables, $n = 7$

Fundamental dimensions in the problem are M, L, T, θ and hence $m = 4$

Number of dimensionless π-terms $= (n - m) = 7 - 4 = 3$

The eqn. (*ii*) may be written as:

$$f_1(\pi_1, \pi_2, \pi_3) = 0$$

We choose h, ρ, D, V as the core group (repeating variables) with unknown exponents. The groups to be formed are now represented as the following π groups.

$$\pi_1 = h^{a_1} . \rho^{b_1} . D^{c_1} . V^{d_1} . \mu$$

$$\pi_2 = h^{a_2} . \rho^{b_2} . D^{c_2} . V^{d_2} . c_p$$

$$\pi_3 = h^{a_3} . \rho^{b_3} . D^{c_3} . V^{d_3} . k$$

π_1–term:

$$M^0L^0T^0 = (MT^{-3}\theta^{-1})^{a_1} . (ML^{-3})^{b_1} . (L)^{c_1} . (LT^{-1})^{d_1} . (ML^{-1}T^{-1})$$

Equating the exponents of M, L, T and θ respectively, we get

For M : $0 = a_1 + b_1 + 1$

For L : $0 = -3b_1 + c_1 + d_1 - 1$

For T : $0 = -3a_1 - d_1 - 1$

For θ : $0 = -a_1$

Solving the above equations, we have

$$a_1 = 0, b_1 = -1, c_1 = -1, d_1 = -1$$

$$\therefore \quad \pi_1 = \rho^{-1} . D^{-1} . V^{-1} . \mu$$

or,

$$\pi_1 = \frac{\mu}{\rho D V}$$

π_2–term:

$$M^0L^0T^0 = (MT^{-3}\theta^{-1})^{a_2} . (ML^{-3})^{b_2} . (L)^{c_2} . (LT^{-1})^{d_2} . (L^2T^{-2}\theta^{-1})$$

For M : $0 = a_2 + b_2$

For L : $0 = -3b_2 + c_2 + d_2 + 2$

For T : $0 = -3a_2 - d_2 - 2$

For θ : $0 = -a_2 - 1$

Solving the above equations, we have

$$a_2 = -1, b_2 = 1, c_2 = 0, d_2 = 1$$

∴ $\pi_2 = h^{-1} . \rho . V . c_p$

or $$\pi_2 = \frac{c_p \rho V}{h}$$

Since dimensions of h and $\frac{k}{D}$ are the same, hence

$$\pi_2 = \frac{c_p \rho VD}{k}$$

π_3–term:

$$\pi_3 = (MT^{-3} \theta^{-1})^{a_3} . (ML^{-3})^{b_3} . (L)^{c_3} . (LT^{-1})^{d_3} . (MLT^{-3} \theta^{-1})$$

Equating the exponents of M, L, T and θ respectively, we get

For M : $0 = a_3 - 3b_3 + 1$

For L : $0 = -3b_3 + c_3 + d_3 + 1$

For T : $0 = -3a_3 - d_3 - 3$

For θ : $0 = -a_3 - 1$

Solving the above equations, we get

$$a_3 = -1, b_3 = 0, c_3 = -1, d_3 = 0$$

∴ $\pi_3 = h^{-1} . D^{-1} . k$

or $$\pi_3 = \frac{k}{hD}$$

According to π–theorem, $\pi_3 = \phi(\pi_1, \pi_2)$

∴ $$\frac{k}{hD} = C \left[\frac{\mu}{\rho DV}\right]^{m'} \left[\frac{c_p \rho DV}{k}\right]^{n'}$$

where m' and n' are constants.

If $m' > n'$, then

$$\frac{k}{hD} = C \left[\frac{\mu}{\rho DV}\right]^{n'} \left[\frac{c_p \rho DV}{k}\right]^{n'} \left[\frac{\mu}{\rho DV}\right]^{m'-n'}$$

$$= C \left[\frac{\mu}{\rho DV}\right]^{m'-n'} \left[\frac{\mu}{\rho DV} . \frac{c_p \rho DV}{k}\right]^{n'}$$

$$= C \left[\frac{\mu}{\rho DV}\right]^{m'-n'} \left[\frac{\mu c_p}{k}\right]^{n'}$$

or $$\frac{hD}{k} = C \left[\frac{\rho DV}{\mu}\right]^{m} \left[\frac{\mu c_p}{k}\right]^{n}$$

or $$Nu = C\,(Re)^m\,(Pr)^n \qquad ...(6.8)$$

where C, m and n are constants and evaluated experimentally

$$\left[\begin{array}{l} \text{where } Nu = \text{Nusselt number} = \dfrac{hD}{k} \\ Re = \text{Reynolds number} = \dfrac{\rho DV}{\mu} \\ Pr = \text{Prandtl number} = \dfrac{\mu c_p}{k} \end{array}\right]$$

It is worth noting that if V, m, ρ, c_p were chosen as the core group (repeating variables), then the analysis would have yielded the following non-dimensional groups:

$$Re = \frac{\rho VD}{\mu};\ Pr = \frac{\mu c_p}{k};\ St = \frac{h}{\rho V c_p}$$

(where, St = Stanton number)

So, another form of correlating heat transfer data is

$$St = \phi (Re, Pr) \qquad \text{...(6.9)}$$

6.6. DIMENSIONAL ANALYSIS APPLIED TO NATURAL OR FREE CONVECTION HEAT TRANSFER

The heat transfer coefficient in case of natural or free convection, like forced convection heat transfer coefficient, depends upon the variables V, ρ, k, μ, c_p and L or D. Since the fluid circulation in free convection is owing to difference in density between the various fluid layers due to temperature gradient and not by external agency, therefore, velocity V is no longer an independent variable but depends upon the following factors:

(*i*) Δt *i.e.*, the difference of temperatures between the heated surface and the undisturbed fluid.

(*ii*) β *i.e.*, coefficient of volume expansion of the fluid.

(*iii*) g *i.e.*, acceleration due to gravity.

($\beta\, g\, \Delta t$ is considered as one physical factor.)

Thus heat transfer coefficient 'h' may be expressed as follows:

$$h = f(\rho, L, \mu, c_p, k, \beta\, g\, \Delta t) \qquad \text{...}(i)$$

$$f_1 (\rho, L, \mu, k, h, c_p, \beta\, g\, \Delta t) \qquad \text{...}(ii)$$

[The parameter ($\beta g\, \Delta t$) represents the buoyant force and has the dimensions of LT^{-2}.]

Total number of variables, $n = 7$

Fundamental dimensions in the problem are M, L, T, θ and hence $m = 4$

Number of dimensionless π-terms = $(n - m) = 7 - 4 = 3$

The equation (*ii*) may be written as:

$$f_1 (\pi_1, \pi_2, \pi_3) = 3$$

We choose ρ, L, μ and k as the core group (repeating variables) with unknown exponents. The groups to be formed are now represented as the following π groups.

$$\pi_1 = \rho^{a_1} . L^{b_1} . \mu^{c_1} . k^{d_1} . h$$

$$\pi_2 = \rho^{a_2} . L^{b_2} . \mu^{c_2} . k^{d_2} . c_p$$

$$\pi_3 = \rho^{a_3} . L^{b_3} . \mu^{c_3} . k^{d_3} . \beta g\, \Delta t$$

π_1-term:

$$M^0 L^0 T^0 \theta^0 = (ML^{-3})^{a_1} . (L)^{b_1} . (ML^{-1} T^{-1})^{c_1} . (MLT^{-3} \theta^{-1})^{d_1} . (ML^{-3} \theta^{-1})$$

Equating the exponents of M, L, T and θ respectively, we get

For M : $0 = a_1 + c_1 + d_1 + 1$

For L : $0 = -3a_1 + b_1 - c_1 + d_1$

For T : $0 = -c_1 - 3d_1 - 3$

For θ : $0 = -d_1 - 1$

Solving the above equations, we get

$$a_1 = 0, b_1 = 1, c_1 = 0, d_1 = -1$$

$$\therefore \quad \pi_1 = Lk^{-1} h \text{ or } \pi_1 = \frac{hL}{k}$$

π_2-term:

$$M^0 L^0 T^0 \theta^0 = (ML^{-3})^{a_2} . (L)^{b_2} . (ML^{-1} T^{-1})^{c_2} . (MLT^{-3} \theta^{-1})^{d_2} . (L^2 T^{-2} \theta^{-1})$$

Equating the exponents of M, L, T, θ respectively, we get

For M : $0 = a_2 + c_2 + d_2$

For L : $0 = -3a_2 + b_2 - c_2 + d_2 + 2$

For T : $0 = -c_2 - 3d_2 - 2$

For θ : $0 = -d_2 - 1$

Solving the above equations, we get

$$a_2 = 0, b_2 = 0, c_2 = 1, d_2 = -1$$

$$\therefore \quad \pi_2 = \mu . k^{-1} . c_p \quad \text{or} \quad \pi_2 = \frac{\mu c_p}{k}$$

π_3–term:

$$M^0 L^0 T^0 \theta^0 = (ML^{-3})^{a_3} . (L)^{b_3} . (ML^{-1} T^{-1})^{c_3} . (MLT^{-3} \theta^{-1})^{d_3} . (LT^{-2})$$

Equating the exponents of M, L, T, θ respectively, we get

For M : $0 = a_3 + c_3 + d_3$

For L : $0 = -3a_3 + b_3 - c_3 + d_3 + 1$

For T : $0 = -c_3 - 3d_3 - 2$

For θ : $0 = -d_3$

Solving the above equations, we get

$$a_3 = 2, b_3 = 3, c_3 = -2, d_3 = 0$$

$$\therefore \quad \pi_3 = \rho^2 . L^3 . \mu^{-2} . (\beta g \Delta t)$$

or,
$$\pi_3 = \frac{(\beta g \Delta t)\rho^2 L^3}{\mu^2} = \frac{(\beta g \Delta t)L^3}{v^2} \quad \text{...(6.10)}$$

or, $\text{Nu} = \phi (Pr) (Gr)$

or, $\text{Nu} = C (Pr)^n (Gr)^m$ (where Gr = Grashoff number) ...(6.11)

Here C, n and m are constants and may be evaluated experimentally.

6.7. ADVANTAGES AND LIMITATIONS OF DIMENSIONAL ANALYSIS

Advantages :

1. It expresses the functional relationship between the variables in dimensionless terms.
2. By the proper selection of variables, the dimensionless parameters can be used to make certain logical deductions about the problem.
3. Design curves, by the use of dimensional analysis, can be developed from the experimental data or direct solution of the problem.