Example No:- 3

- Fluid of density ρ and viscosity μ flows at an average velocity V through a circular pipe diameter d. show by dimensional analysis, that the shear stress of the pipe wall.

$$\tau_o = \rho V^2 f\left[\frac{\rho V d}{\mu}\right]$$

- **Soln.:** The relationship between dependant and independent variable may be expressed as:

  $F = f(d, v, \mu, \rho, \tau_o)$

- Dimension of the variable involved :

Sr. No.	Variable	Symbol	Dimension
1.	Shear stress	τ_o	$ML^{-1}T^{-2}$
2.	Viscosity	μ	$ML^{-1}T^{-1}$
3.	Density	ρ	ML^{-3}

Number of variables $n = 5$

Number of fundamental dimension $m = 3$

Number of dimensionless π term $(n - m) = 2$

$$f1\,(\pi 1, \pi 2) = 0 \qquad \text{............}(1)$$

Selecting the repeating variable as **d, V, ρ**

Each π term contain $m+1 = 3+1 = 4$ variables.

- By using Buckingham's theorem,

$$\pi_1 = d^{a1} \cdot V^{b1} \cdot \rho^{c1} \cdot \tau_o$$

$$\pi_2 = d^{a2} \cdot V^{b2} \cdot \rho^{c2} \cdot \mu$$

- Solve the π equation by the principle of dimension homogeneity:
- **For π_1 term:**

$$\pi_1 = d^{a1} \cdot V^{b1} \cdot \rho^{c1} \cdot \tau_o$$

$$M^0L^0T^0 = L^{a_1}\cdot(LT^{-1})^{b_1}\cdot(ML^{-3})^{c_1}\cdot(ML^{-1}T^{-2})$$

$$\therefore c_1 = -1$$

$$M: 0 = c_1 + 1 \qquad \therefore a_1 = 0$$

$$L: 0 = a_1 - 3c_1 - 1 + b_1 \qquad \therefore b_1 = -2$$

$$T: 0 = -b_1 - 2$$

$$\pi_1 = d_1^{\,0}\cdot V^{-2}\cdot\rho^{-1}\cdot\tau_o = \frac{\tau_o}{V^2\rho}$$

- For π_2 term:

$$M^0L^0T^0 = L^{a_2}\cdot(LT^{-1})^{b_2}\cdot(ML^{-3})^{c_2}\cdot(ML^{-1}T^{-1})$$

$$M: 0 = c_2 + 1 \qquad \therefore c_2 = -1$$

$$L: 0 = a_2 - 3c_2 + b_2 - 1 \qquad \therefore a_2 = -1$$

$$T: 0 = -b_2 - 1 \qquad \therefore b_2 = -1$$

$$\pi_2 = d^{-1}\cdot V^{-1}\cdot\rho^{-1}\cdot\mu = \frac{\mu}{dV\rho} = \frac{1}{\pi} = {dV\rho}/{\mu}$$

- Substituting the value of π_1, π_2 in equation (1),

$$f_1\left(\frac{\tau_o}{V^2\rho}, \frac{dV\rho}{\mu}\right)$$

$$\frac{\tau_o}{V^2\rho} = f_1\left(\frac{dV\rho}{\mu}\right)$$

$$\boxed{\tau_o = V^2\rho f_1\left(\frac{dV\rho}{\mu}\right)}$$

....Ans.

Example No:- 4

This example is elementary, but demonstrates the general procedure: Suppose a car is driving at 100 km/hour; how long does it take it to go 200 km? This question has two fundamental physical units: time t and length , and three dimensional variables: distance D, time taken T, and velocity V. Thus there are $3 - 2 = 1$ dimensionless quantity. The units of the dimensional quantities are:

$$D \sim \ell, \; T \sim t, \; V \sim \ell/t.$$

The dimensional matrix is:

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

The rows correspond to the dimensions , and t, and the columns to the dimensional variables D, T, V. For instance, the 3rd column, $(1, -1)$, states that the V (velocity) variable has units of

$$\ell^1 t^{-1} = \ell/t$$

- For a dimensionless constant $\pi = D^{a_1} T^{a_2} V^{a_3}$ we are looking for a vector $a = [a_1, a_2, a_3]$ such that the matrix product of M on a yields the zero vector [0,0]. In linear algebra, this vector is known as the kernel of the dimensional matrix, and it spans the nullspace of the dimensional matrix, which in this particular case is one-dimensional. The dimensional matrix as written above is in reduced row echelon form, so one can read off a kernel vector within a multiplicative constant:

$$a = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

If the dimensional matrix were not already reduced, one could perform Gauss-Jordan elimination on the dimensional matrix to more easily determine the kernel. It follows that the dimensionless constant may be written:

$$\pi = D^{-1}T^{1}V^{1} = TV/D$$

or, in dimensional terms

$$\pi \sim (\ell)^{-1}(t)^{1}(\ell/t)^{1} \sim 1$$

- Since the kernel is only defined to within a multiplicative constant, if the above dimensionless constant is raised to any arbitrary power, it will yield another equivalent dimensionless constant.
- Dimensional analysis has thus provided a general equation relating the three physical variables

$$f(\pi) = 0$$

which may be written

$$T = \frac{CD}{V}$$

- where C is one of a set of constants, such that $C = f^{-1}(0)$. The actual relationship between the three variables is simply $D = VT$ so that the actual dimensionless equation ($f(\pi) = 0$) is written:

$$f(\pi) = \pi - 1 = VT/D - 1 = 0$$

- In other words, there is only one value of C and it is unity. The fact that there is only a single value of C and that it is equal to unity is a level of detail not provided by the technique of dimensional analysis.

Example No:- 5

- We wish to determine the period T of small oscillations in a simple pendulum. It will be assumed that it is a function of the length L, the mass M, and the acceleration due to gravity on the surface of the Earth g, which has dimensions of length divided by time squared. The model is of the form

$$f(T, M, L, g) = 0.$$

- (Note that it is written as a relation, not as a function: T isn't written here as a function of M, L, and g.)
- There are 3 fundamental physical dimensions in this equation: time t, mass m, and length l, and 4 dimensional variables, T, M, L, and g. Thus we need only $4 - 3 = 1$ dimensionless parameter, denoted π, and the model can be re-expressed as

where **π** is given by

$$f(\pi) = 0$$

for some values of $a_1, \ldots, a_4$.
The dimensions of the dimensional quantities are:

$$\pi = T^{a_1} M^{a_2} L^{a_3} g^{a_4}$$

for some values of $a_1, \ldots, a_4$.
The dimensions of the dimensional quantities are:

$$T = t, M = m, L = \ell, g = \ell/t^2.$$

The dimensional matrix is:

$$M = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

- (The rows correspond to the dimensions t, m, and ℓ, and the columns to the dimensional variables T, M, L and g. For instance, the 4th column, (−2, 0, 1), states that the gvariable has dimensions of $t^{-2}m^0\ell^1$.
- We are looking for a kernel vector $a = [a_1, a_2, a_3, a_4]$ such that the matrix product of M on a yields the zero vector [0,0,0]. The dimensional matrix as written above is in reduced row echelon form, so one can read off a kernel vector within a multiplicative constant:

$$a = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

- Were it not already reduced, one could perform Gauss-Jordan elimination on the dimensional matrix to more easily determine the kernel. It follows that the dimensionless constant may be written:

$$\pi = T^2 M^0 L^{-1} g^1$$
$$= gT^2/L$$

- In fundamental terms:

$$\pi = (t)^2 (m)^0 (\ell)^{-1} (\ell/t^2)^1 = 1$$

- which is dimensionless. Since the kernel is only defined to within a multiplicative constant, if the above dimensionless constant is raised to any arbitrary power, it will yield another equivalent dimensionless constant
- This example is easy because three of the dimensional quantities are fundamental units, so the last (g) is a combination of the previous. Note that if a_2 were non-zero there would be no way to cancel the M value—therefore a_2 *must* be zero. Dimensional analysis has allowed us to conclude that the period of the pendulum is not a function of its mass. (In the 3D space of powers of mass, time, and distance, we can say that the vector for mass is linearly independent from the vectors for the three other variables. Up to a scaling factor $\vec{g} - 2\vec{T} - \vec{L}$ is the only nontrivial way to construct a vector of a dimensionless parameter.)
- The model can now be expressed as

$$f(gT^2/L) = 0.$$