6. Phasors in circuit analysis

We are now in a position to summarise the method of analysis of AC circuits.

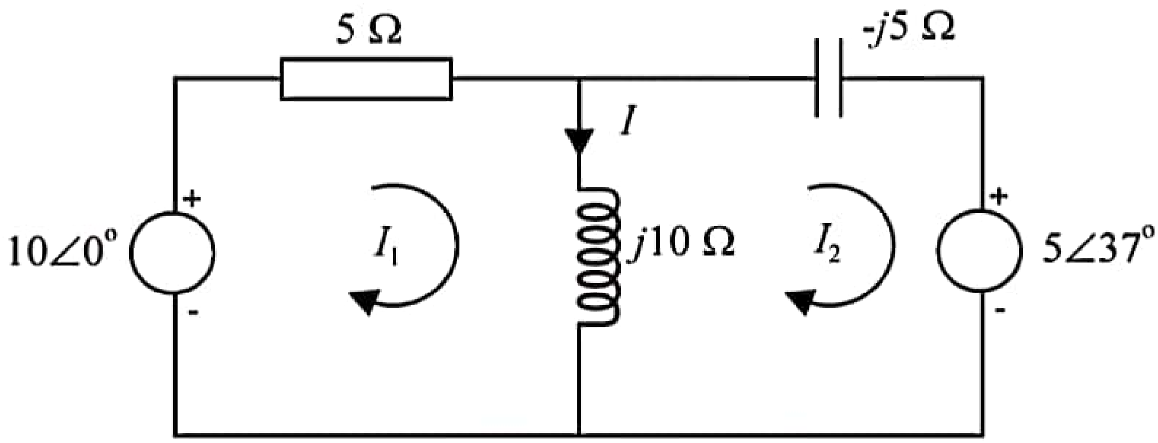
(i) We include all reactances as imaginary quantities $j\omega L\left(=jX_L\right)$ for an inductor and $1/j\omega C\left(=-jX_c\right)$ for a capacitor.

(ii) All voltages and currents are represented by phasors, which usually have rms magnitude, and one is chosen as a reference with zero phase angle.

(iii) All calculations are carried out in complex notation.

(iv) The magnitude and phase of, say, the current is obtained as $|I|\exp j\phi$. This can, if necessary, be converted back into a time varying expression $\sqrt{2}|I|\cos(\omega t+\phi)$.

Suppose we wish to find the current flowing through the inductor in the circuit below



The reactances have been calculated and marked on the diagram. The left hand voltage source has been chosen as reference and provides 10*V* rms. The right hand source produces 5V rms but at a phase angle of 37° with respect to the 10*V* source. If we introduce *phasor loop currents* I_1 and I_2 as shown then we may write *KVL* loop equations as

$$10 = 5I_1 + j10(I_1 - I_2)$$
$$5\exp j37^{\circ} = 4 + j3 = -(I_2 - I_1)j10 - (-j5)I_2$$

where we have noted that $5\exp j37^{\circ} = 5\cos 37^{\circ} + j5\sin 37^{\circ} = 4 + j3$. It is routine to solve these simultaneous equations to give

$$I_1 = \frac{(7-j)}{12.5} \quad \text{and} \quad I_2 = \frac{6.5 + j8}{12.5}$$

and hence the current $I = I_1 - I_2$ becomes

$$I = \frac{0.5 \quad j9}{12.5} = 0.72\angle -86.8^{o} = 0.72\exp -j86.8^{o}$$

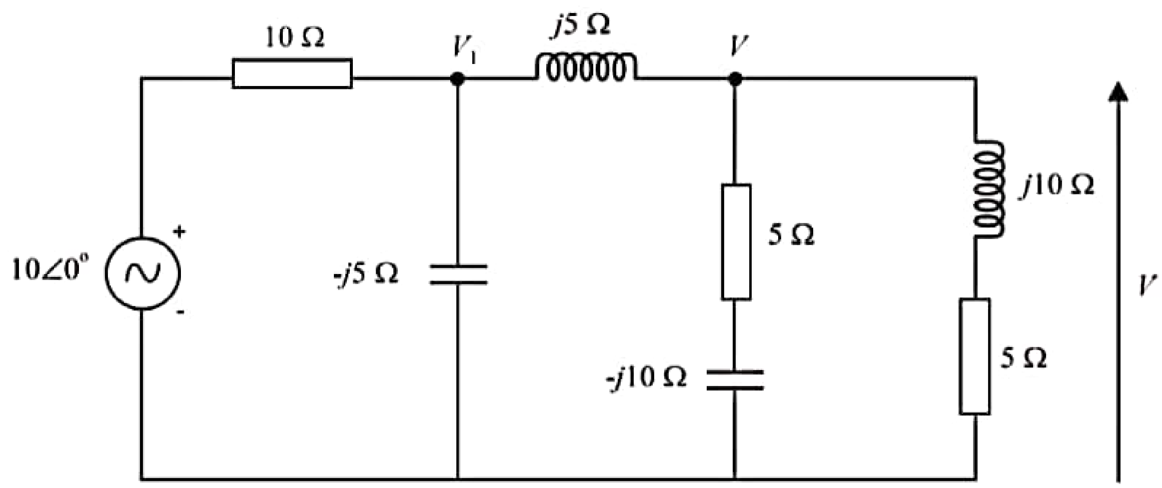
Since rms values are involved, if we want to convert this into a function of time we must multiply by $\sqrt{2}$ to obtain the peak value. Thus

$$i(t) = 1.02\cos\left(\omega t - 86.8^{o}\right)$$

In our example we do not know the value of ω but it was accounted for in the value of the reactances. Since everything is linear and the sources are independent it would be a good exercise for you to check this result by using the principle of *superposition*.

We have used mesh or loop analysis in our examples so far. It is, of course equally appropriate to use node-voltage analysis if that looks like an easier way to solve the problem.

As an example let's suppose we would like to find the voltage *V* in the circuit below where the reactances have been calculated corresponding to the frequency, ω, of the source



It's probably as easy as anything to introduce two *phasor* node voltages V_1 and V. The two node voltage relationships may be written as

$$\frac{V_1 - 10}{10} + \frac{V_1 - V}{j5} + \frac{V_1 - 0}{-j5} = 0$$

and

$$\frac{V - V_1}{j5} + \frac{V - 0}{5 - j10} + \frac{V - 0}{5 + j10} = 0$$

We note that in writing these equations no thought was given to whether currents flowing into or out of the nodes were being considered. As in the DC case it is merely necessary to be consistent. It is now straightforward to solve these two simultaneous equations to yield $V_1 = \sqrt{10}\angle -71.6^o$ or, if the time domain result is required, remembering that the voltage supply is 10V rms then $v_1(t) = \sqrt{20}\cos(\omega t - 71.6^o)$.

7. Combining Impedances

As we have seen before the ratio of the voltage to the current phasors is in general a complex quantity, Z, which generalises Ohms law, in terms of phasors, to

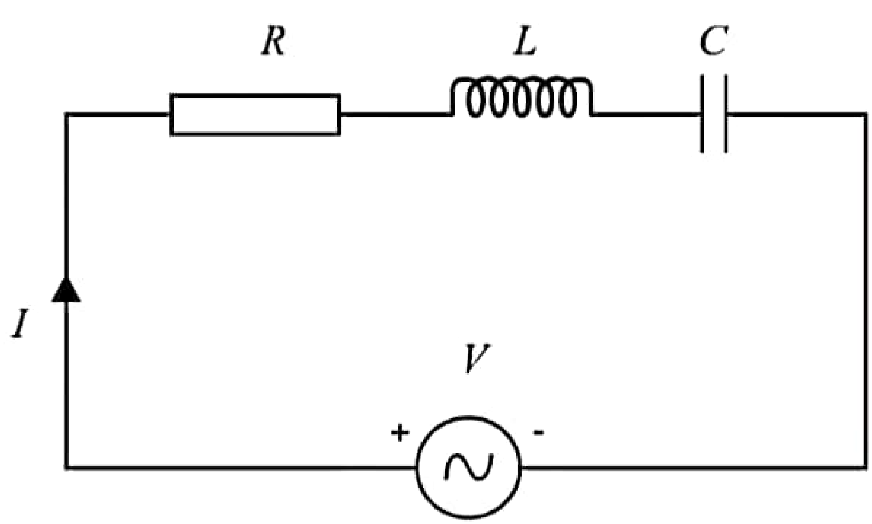
$$V = Z I$$

where Z in general takes the form

$$Z = R_e + j X_e$$

where the overall effect is equivalent to a resistance, R_e, in series with a reactance X_e. If X_e is positive the effective reactance is inductive whereas negative values suggest that the effective reactance is capacitative.

Consider the circuit below

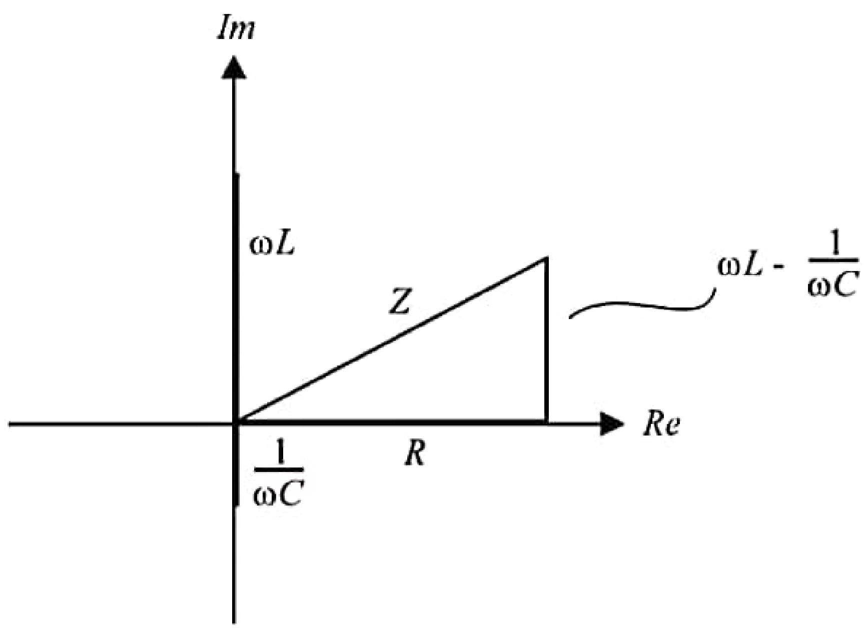


$$V = \left(R + j\omega L + \frac{1}{j\omega C} \right) I$$

Thus the combined impedance Z is given by

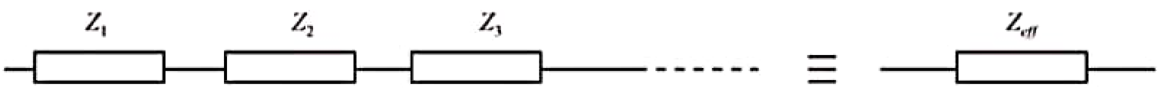
$$Z = R + j\left(\omega L - \frac{1}{\omega C} \right) = R + jX$$

This may be visualised on an Argand or phasor diagram



We note that the reactance may be positive or negative according to the relative values of ωL and $1/\omega C$. Indeed at a frequency $\omega = \sqrt{LC}$ we see that $X = 0$ and that the impedance is purely resistive. We will return to this point later.

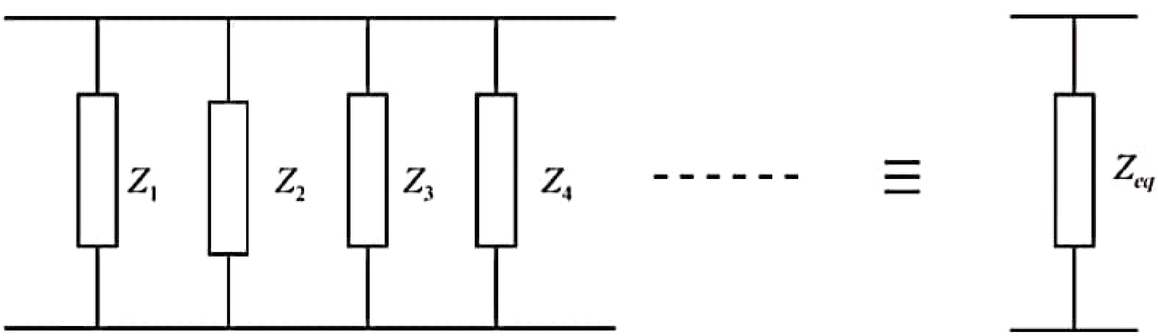
It is straightforward to show, and hopefully intuitive, that all the DC rules for combining resistances in series and parallel carry over to impedances. Thus if we have *n* elements in series, $Z_1, Z_2, Z_3 \ldots Z_n$



Where

$$Z_{eff} = Z_1 + Z_2 + Z_3 + \ldots Z_n = \sum_{i=1}^{n} Z_i$$

and similarly for parallel elements



$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \ldots = \sum_{i=1}^{N} \frac{1}{Z_i}$$

We note that the inverse of impedance, *Z*, is known as **admittance**, *Y*. Thus, as in the DC case it is sometimes more convenient to write

$$Y_{eq} = \sum_{i=1}^{n} Y_i$$

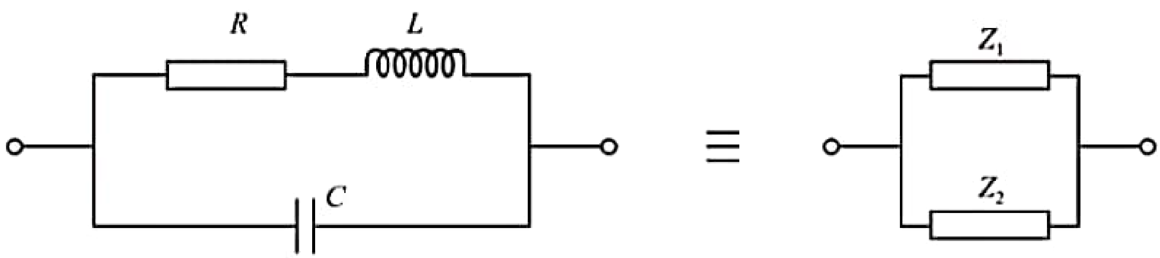
and finally since Y is also a complex number it may be written

$$Y = G + jB$$

where G is a **conductance** and B is known as the **susceptance.**

Example

Find the equivalent impedance of the circuit below



Using the usual combination rules gives

$$Z = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(R + j\omega L)\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}}$$

Which we can simplify to

$$Z = \frac{R + j\omega L}{1 - \omega^2 LC + j\omega CR}$$

8. Operations on phasors

We have just introduced a method of analysing AC circuits in terms of complex currents and voltages. This method inevitably involves the manipulation of complex phasor quantities and so we list below the results for manipulating these quantities which are, course, simply the standard rules for complex numbers. Sometimes it is easier to use the $a+jb$ notation and sometimes the $r\exp j\theta = r\angle\theta$ notation is easiest. We summarise below the important relationships.

Addition and Subtraction

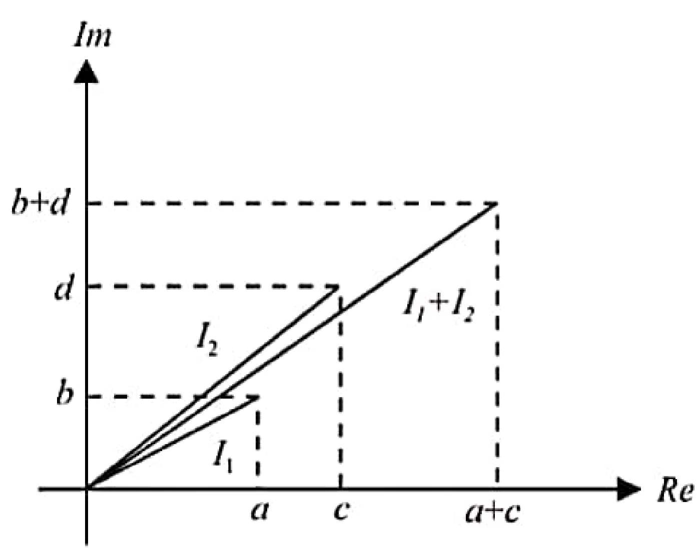
If

$$I_1 = a + jb \text{ and } I_2 = c + jd$$

then

$$I_1 \pm I_2 = a \pm c + j(b \pm d)$$

where the real and imaginary parts add/subtract



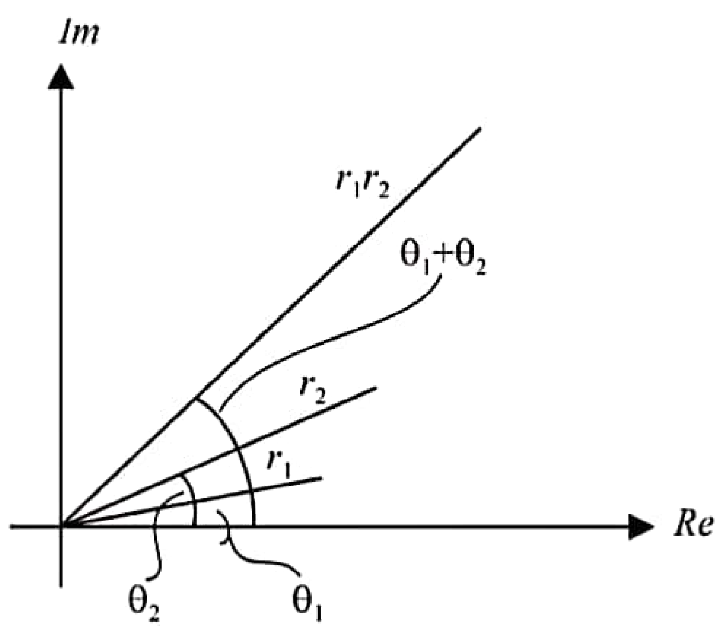
Multiplication

Here it is easiest *by far* to use the $r \angle\theta$ rotation.

If $\qquad I_1 = r_1 \angle\theta_1 = r_1 \exp j\theta_1$ and $I_2 = r_2 \exp j\theta_2$

then $\qquad I_1 I_2 = r_1 r_2 \exp j(\theta_1 + \theta_2) = r_1 r_2 \angle\theta_1 + \theta_2$

when we see the amplitudes multiply and the arguments add



For **division** we have

$$\frac{I_1}{I_2} = \frac{r_1 \exp j\theta_1}{r_2 \exp j\theta_2} = \frac{r_1}{r_2} \exp j(\theta_1 - \theta_2) = \frac{r_1}{r_2} \angle\theta_1 - \theta_2$$

the amplitudes divide and the arguments subtract.

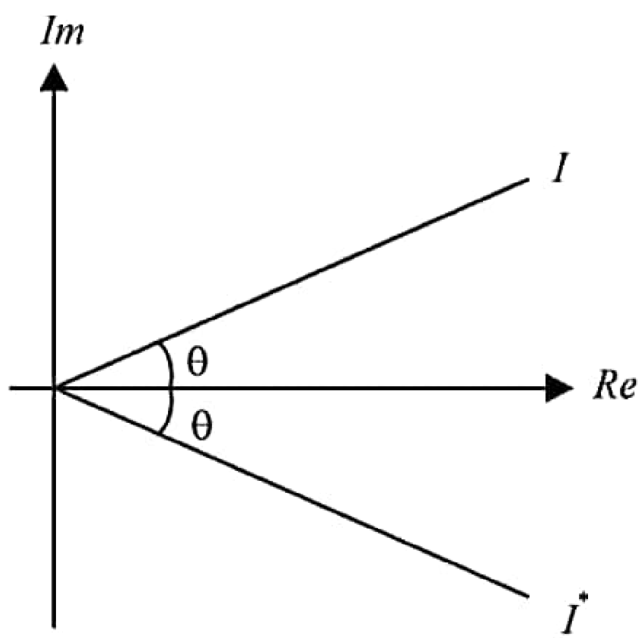
Complex conjugates also appear.

If

$$I = a + j\,b = r \exp j\,\theta = r\,\angle\theta$$

then the complex conjugate I^*, is given by

$$I^* = a - j\,b = r \exp - j\,\theta = r\,\angle -\theta$$



from which we see

$$I + I^* = 2\mathrm{Re}\{I\}\,;\; I - I^* = 2j\,\mathrm{Im}\{I\}$$

where Re{ } denotes the real part the Im { } denotes the imaginary part.

Rationalising

We are often confronted with expressions of the form

$$\frac{a+jb}{c+jd}$$

And sometimes we wish to rationalise them. We do this in one of two ways. The first is to multiply top and bottom by $c-jd$. This gives

$$\frac{a+jb}{c+jd}=\frac{a+jb}{c+jd}\;\frac{c-jd}{c-jd}=\left(\frac{ac+bd}{c^2+d^2}\right)+j\left(\frac{bc-ad}{c^2+d^2}\right)$$

Alternatively, we can write $a+jb$ as $r_1\exp j\theta_1$ with $r_1=\sqrt{a^2+b^2}$ and $\tan\theta_1=b/a$. Similarly $c+jd$ may be written as $r_2\exp j\theta_2$ and hence

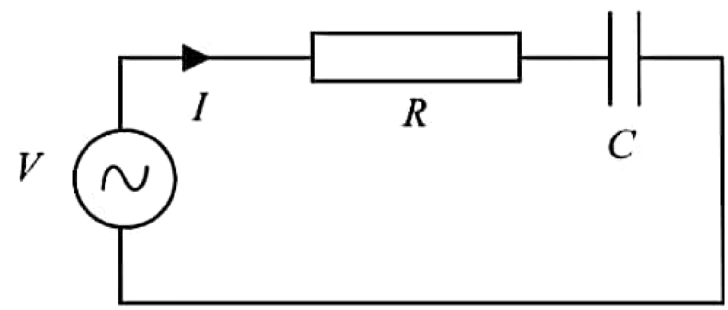
$$\frac{a+jb}{c+jd}=\frac{r_1\exp j\theta_1}{r_2\exp j\theta_2}=\frac{r_1}{r_2}\exp j(\theta_1-\theta_2)=\frac{r_1}{r_2}\angle\theta_1-\theta_2$$

where the amplitudes divide and the arguments subtract.

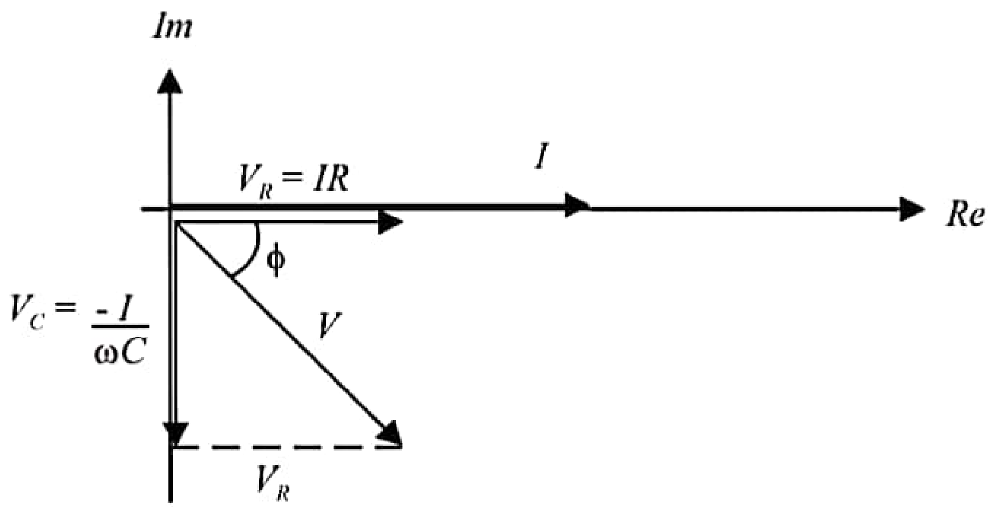
There is no golden rule as to which approach to take – it is determined by the problem at hand. However, if you do not have a pressing need to rationalise the expression, we will see some quite good reasons why we may often prefer to stick with the factorised form and not rationalise at all.

9. Phasor diagrams

Although direct calculation is easily carried out using phasors it is sometimes useful to use a phasor (Argand) diagram to show the relationships between say a voltage and current phasor graphically. In this way it is easy to see their relative amplitudes and phases and hence gain quick insight into how the circuit operates. As a simple example consider the circuit below

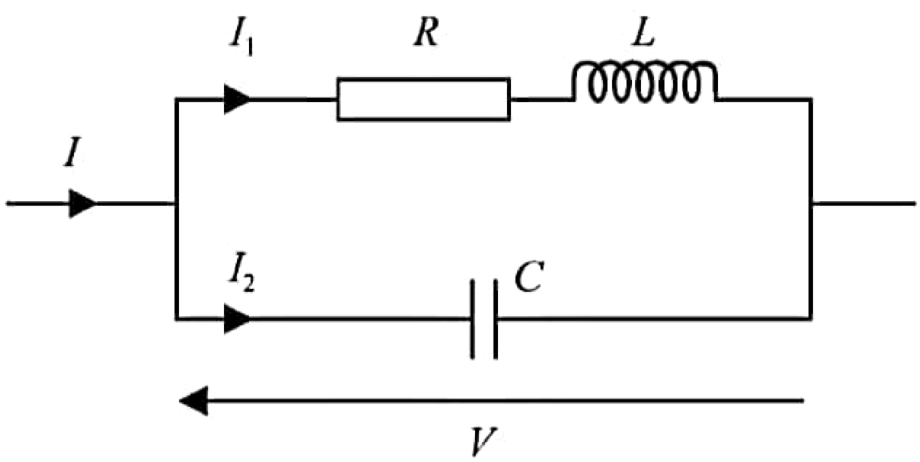


Since the current, I, flows through both elements it is sensible to choose this as the *reference* phasor. Having made this choice the voltage drop across the resistor, $V_R = IR$, whereas that across the capacitor, $V_c = I/j\omega C = -j/\omega C I$. The sum of these voltages must equal V. The phasor diagram is easily drawn as



from which it is clear that the voltage lags the current by an angle ϕ. The angle ϕ may be obtained from the diagram as $\phi = \tan^{-1}(1/\omega CR)$.

Let's consider another example

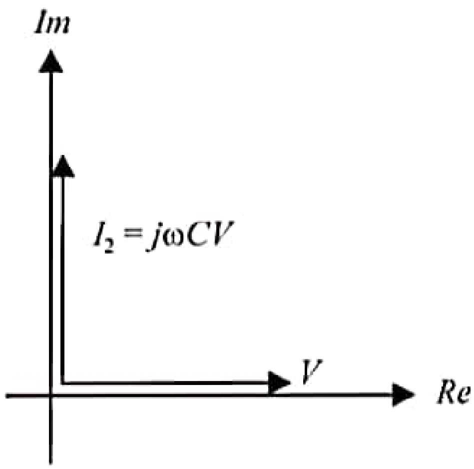


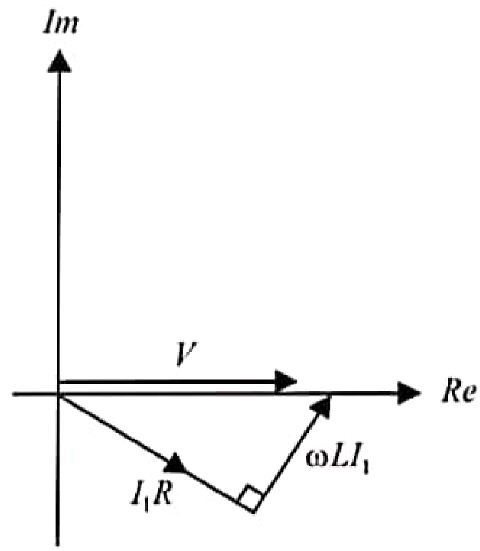
We could obtain the relationship between *I* and *V* by using the equivalent impedance derived earlier. However we will use a phasor diagram to show the various currents and voltages which appear across the various components. Since the voltage, *V*, is the same across each arm it is sensible to choose this as the reference phasor.

The relationships are

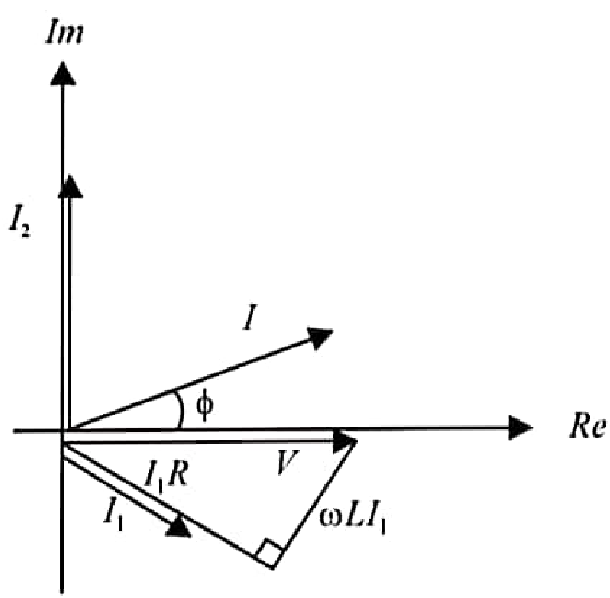
$$I_2 = j\omega CV \quad V = I_1 R + j\omega L I_1 \quad \text{and} \quad I = I_1 + I_2$$

The two phasor diagrams are





or, combining onto a single diagram



where ϕ denotes the phase angle between *V* and I. In the diagram above *I* leads *V* by ϕ.