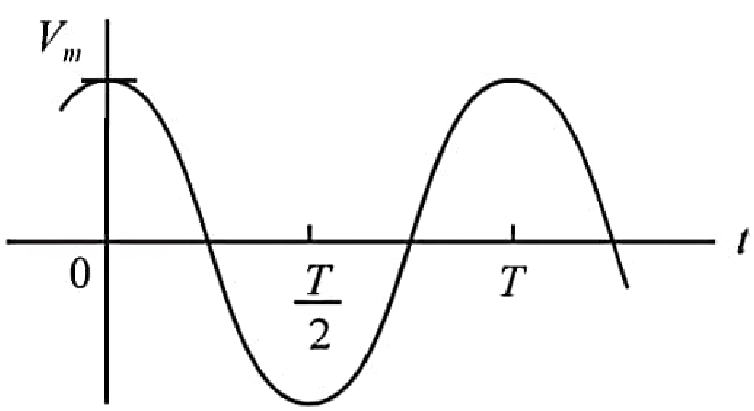
AC Circuits

1. Basic Ideas

Our development of the principles of circuit analysis in Circuit Analysis I was in terms of DC circuits in which the currents and voltages were constant and so did not vary with time. We will now extend this analysis to consider time varying currents and voltages. In our initial discussions we will limit ourselves to sinusoidal functions. We choose this special case because, as you have now learnt in P1, it allows us to make use of some very powerful and helpful mathematical techniques. It is a common waveform in nature and it is easy to generate in the lab. However as you have also learnt in P1, *any* waveform can be expressed as a weighted superposition of sinusoids of different frequencies and hence if we analyse a *linear* circuit for sinusoidal functions we can, by appropriate superposition, handle any function of time.

Let's begin by considering a sinusoidal variation in voltage

$$v = V_m \cos \omega t$$



in which ω is the *angular frequency* and is measured in radians/second. Since the angle ωt must change by 2π radians in the course of one period, T, it follows that

$$\omega T = 2\pi$$

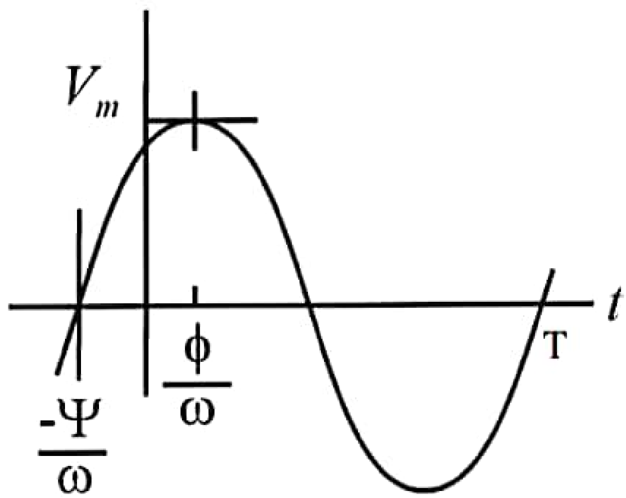
However the time period $T = \frac{1}{f}$ where f is the *frequency* measured in *Hertz*. Thus

$$\omega = \frac{2\pi}{T} = 2\pi f$$

This is a simple and very important relationship. We naturally measure frequency in Hz – the mains frequency in the UK is 50Hz – and it is easy to measure the time period, $T(= 1/f)$ from an oscilloscope screen. However as we will soon see, it is mathematically more convenient to work in terms of the angular frequency ω. Mistakes may be easily made because in practice the word frequency is commonly used to refer to both ω and f. It is important in calculations to make sure that if ω appears, then the correct value for $f = 50$ Hz, say, is $\omega = 100\pi$ rads/sec. A simple point to labour I admit, but if I had a pound for every time someone forgets and substitutes $\omega = 50$ . . . . !!

In our example above, $v = V_m \cos \omega t$, it was convenient that $v = V_m$ at $t = 0$. In general this will not be the case and the waveform will have an arbitrary relationship to the origin $t = 0$ or, equivalently the origin may have been chosen arbitrarily and the voltage, say, may be written in terms of a *phase angle*, ϕ, as

$$v = V_m \cos(\omega t - \phi)$$



Alternatively, in terms of a different phase angle, ψ, the *same* waveform can be written

$$v = V_m \sin(\omega t + \psi)$$

where

$$\psi = \pi/2 - \phi$$

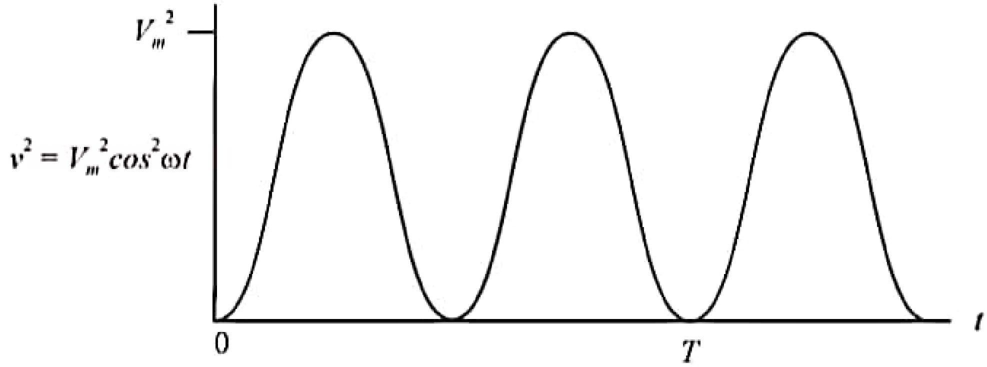
The phase difference between two sinusoids is almost always measured in angle rather than time and of course one cycle (*i.e.* one period) corresponds to 2π or 360°. Thus we might say that the waveform above is *out of phase* with the earlier sinusoid by ϕ. When $\phi = \pm\pi/2$ we say that the two sinusoids are said to be in *quadrature*. When $\phi = \pi$ the sinusoids are in opposite phase or in *antiphase*.

2. RMS Values

We refer to the maximum value of the sinusoid, V_m, as the "peak" value. On the other hand, if we are looking at the waveform on an oscilloscope, it is usually easier to measure the "peak-to-peak" value $2V_m$, *i.e.* from the bottom to the top. However, you will notice that most meters are calibrated to measure the *root-mean-square* or *rms* value. This is found, as the name suggests, for a particular function, *f*, by squaring the function, averaging over a period and taking the (positive) square root of the average. Thus the rms value of any function $f(x)$, over the interval x to $x+X$, where X denotes the period is

$$f_{rms} = \sqrt{\frac{1}{X}\int_{x}^{x+X} f^2(y)\,dy}$$

For our sinusoidal function $v = V_m \cos \omega t$



The average of the square is given by

$$\frac{1}{T}\int_0^T V_m^2 \cos^2 \omega t \, dt$$

where the time period $T = 2\pi/\omega$. At this point it's probably easiest to change variables to $\theta = \omega t$ and to write $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$. Thus the mean square value becomes

$$\frac{1}{2}\frac{V_m^2}{2\pi}\int_0^{2\pi}(1 + \cos 2\theta)d\theta = \frac{V_m^2}{2}$$

The root mean square value, which is simply the positive square root of this, may be written as

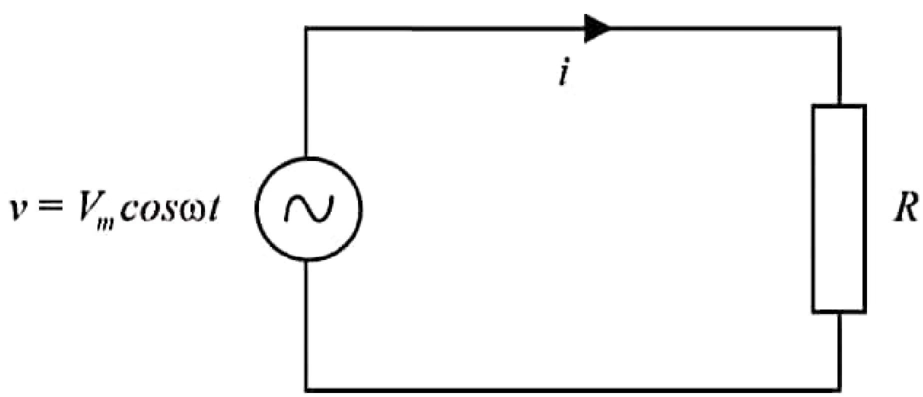
$$V_{rms} = V_m/\surd 2 \approx 0.7\ V_m.$$

Since we nearly always use rms values in our AC analysis, we assume rms quantities unless told otherwise so by convention we just call it *V* as in:

$$V = V_m/\sqrt{2}.$$

So, for example, when we say that the UK mains voltage is 230*V* what we are really saying is that the rms value 230V. Its *peak* or maximum value is actually $230\sqrt{2} \approx 325$ V.

To see the real importance of the rms value let's calculate the power dissipated in a resistor.



Here the current is given by $i = v/R$

$$i = \frac{V_m}{R}\cos\omega t = I_m \cos\omega t$$

where $I_m = V_m/R$ and the power, $p = vi$, is given by

$$p = \frac{V_m^2}{R}\cos^2 \omega t$$

If we want to calculate the average power dissipated over a cycle we must integrate from $\omega t = 0$ to $\omega t = \omega T = 2\pi$. If we again introduce $\theta = \omega t$, the average power dissipated, P, is given by

$$P = \frac{1}{2\pi} \cdot \frac{V_m^2}{R} \int_0^{2\pi} \cos^2 \theta \, d\theta$$

$$P = \frac{1}{2\pi} \frac{V_m^2}{R} \cdot \frac{1}{2} \int_0^{2\pi} (1 + \cos 2\theta) d\theta$$

$$P = V_m^2 / 2R$$

If we now introduce the rms value of the voltage $V = V_m / \sqrt{2}$ then the average power dissipated may be written as

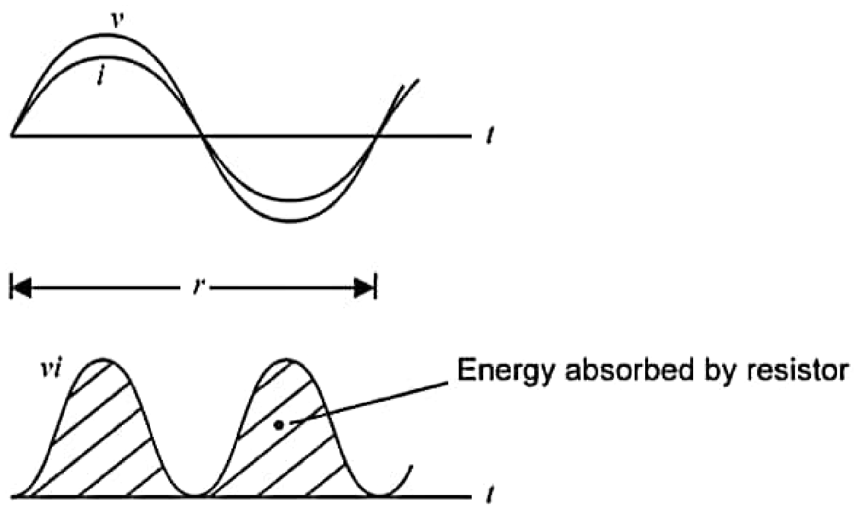
$$P = V^2 / R$$

Indeed if the rms value of the current $I = I_m / \sqrt{2}$ is also introduced then

$$P = V^2 / R = I^2 R$$

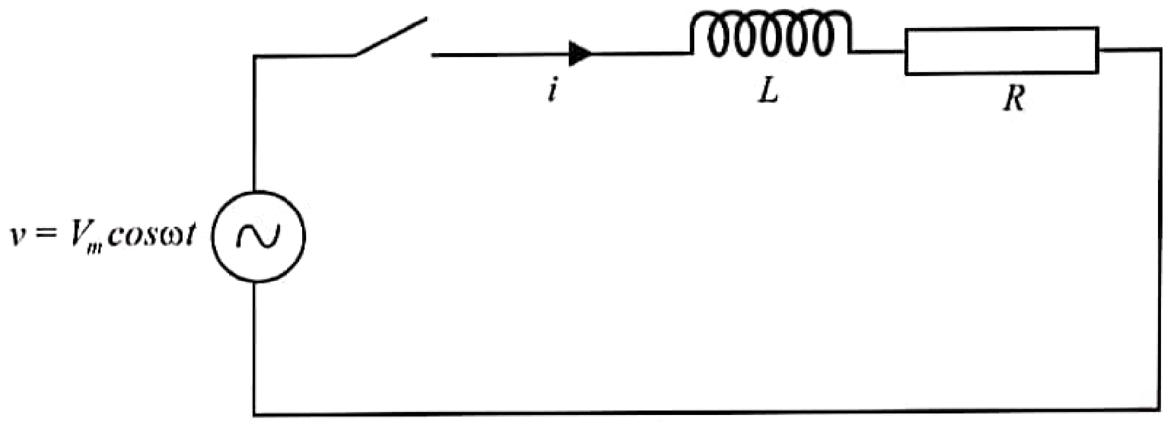
which is *exactly* the same form of expression we derived for the DC case.

Therefore if we use rms values we can use the same formula for the average power dissipation irrespective of whether the signals are AC or DC.


v
i
t
r
vi
Energy absorbed by resistor
t

3. Circuit analysis with sinusoids

Let us begin by considering the following circuit and try to find an expression for the current, *i*, after the switch is closed.



The Kirchhoff voltage law permits us to write

$$L\frac{di}{dt} + Ri = V_m \cos\omega t$$

This is a linear differential equation, which you know how to solve.
We begin by finding the *complementary function*, from the *homogeneous equation*:

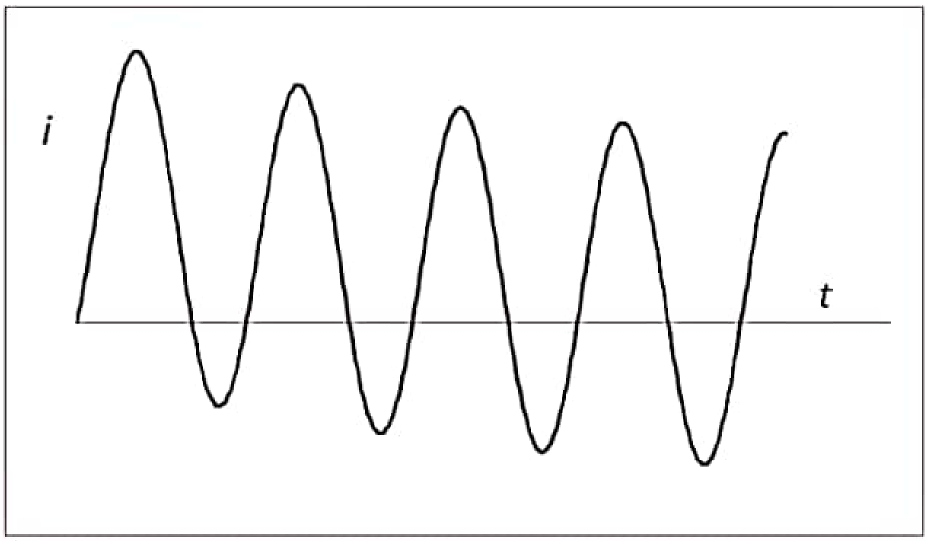
$$L\frac{di}{dt} + Ri = 0$$

which yields the solution:

$$i = A \ \exp-(Rt/L)$$

We now need to find the *particular integral* which, for the sinusoidal "forcing function" $V_m \cos \omega t$, will take the form $B \cos \omega t + C \sin \omega t$. Thus the full solution is given by

$$i(t) = A \exp -(Rt/L) + B \cos \omega t + C \sin \omega t$$



We see that the current consists of a "transient" term, $A \exp -(Rt/L)$, which eventually decays and becomes negligible in comparison with the "steady state" response. The transient response arises because of the sudden opening or closing of a switch but we will concentrate here on the final sinusoidal steady state response. How long do we have to wait for the steady state? If for example $R = 100\Omega$ and $L = 25mH$ then

$R/L = 4 \times 10^{-3} \sec^{-1}$ and so after only $1ms$ $\exp - Rt/L = \exp(-4) = 0.018$ and so any measurements we are likely to make on this circuit will be truly 'steady state' measurements. Thus our solution of interest reduces to

$$i = B\cos\omega t + C\sin\omega t$$

In order to find *B* and *C* we need to substitute this expression back into the governing differential equation to give

$$\omega L\{C\cos\omega t - B\sin\omega t\} + R\{B\cos\omega t + C\sin\omega t\} = V_m\cos\omega t$$

It is now a simple matter to compare coefficients of cosωt and sinωt to obtain expressions for *B* and *C* which lead, after a little algebra, to

$$i = \frac{V_m}{R^2 + (\omega L)^2}\{R\cos\omega t + \omega L\sin\omega t\}$$

If we now introduce the **inductive reactance** $X_L(=\omega L)$ we can write this equation as

$$i = \frac{V_m}{\sqrt{R^2 + X_L^2}}\left\{\frac{R}{\sqrt{R^2 + X_L^2}}\cos\omega t + \frac{X_L}{\sqrt{R^2 + X_L^2}}\sin\omega t\right\}$$

The expression in curly brackets is of the form

$$\cos\phi\cos\omega t + \sin\phi\sin\omega t = \cos(\omega t - \phi)$$

and hence

$$i = \frac{V_m}{\sqrt{R^2 + X_L^2}}\cos(\omega t - \varphi)$$

where

$$\varphi = \tan^{-1}\left(\frac{X_L}{R}\right)$$

Thus we see that the effect of the inductor has been to introduce a phase lag ϕ between the current flowing in the circuit and the voltage source. Similarly the ratio of the maximum voltage to the maximum current is given by $\sqrt{R^2 + X_L^2}$ which since it is a combination of resistance and reactance is given the new name of **impedance**.

It is apparent that we could solve all networks containing combinations of resistors, inductors and capacitors in this way. We would end up with a series of simultaneous equations to solve – just as we did when analysing DC circuits – the problem is that they would be simultaneous *differential* equations which, given the effort we went through to solve one equation in the simple example above, would be very tedious and therefore rather error-prone. Fortunately there is an easier way.

We are saved because the differential equations we have to solve are **linear** and hence the principle of **superposition** applies. This tells us that if a forcing function $v_1(t)$ produces current $i_1(t)$ and a forcing function $v_2(t)$ provides current $i_2(t)$ then $v_1(t)+v_2(t)$ produces $i_1(t)+i_2(t)$. The trick then is to choose a more general forcing function $v(t)=v_1(t)+v_2(t)$ in which, say, $v_1(t)$ corresponds to $V_m \cos \omega t$ *and* which made the differential equation easy to solve. We achieve this with *complex algebra*.

You should know that

$$\exp j\,\omega t = \cos\omega t + j\sin\omega t \;.$$

where j = √(-1),   [electrical engineers like to use i for current]

so let's solve the differential equation with the general forcing function

$$v(t) = V_m \exp j\,\omega t = V_m \cos\omega t + j\,V_m \sin\omega t$$

where

$$v_1(t) = \mathrm{Re}\{v(t)\} = \mathrm{Re}\{V_m \exp j\,\omega t\} = V_m \cos\omega t \;.$$

The solution will be of the form

$$i(t) = I \exp j\omega t$$

where $I = |I|\exp(-j\emptyset)$ will, in general, be a complex number. Then in order to find that part of the full solution corresponding to the real part of the forcing function, $V_m \exp j\,\omega t$ we merely need to find the real part of $i(t)$. Thus

$$\begin{aligned} i_1(t) &= \mathrm{Re}\{I \exp \mathrm{j}\omega t\} = \mathrm{Re}\{|I| \exp \mathrm{j}(\omega t - \varphi)\} \\ &= |I|\cos(\omega t - \varphi) \end{aligned}$$

Let's illustrate this by returning to our previous example where we tried to solve:

$$L\frac{di}{dt}+Ri=V_m\cos\omega t$$

Now, instead, we solve the more general case:

$$L\frac{di}{dt}+Ri=V_m\exp j\omega t$$

and take the real part of the solution. As suggested above an appropriate particular integral is $i=I\exp j\omega t$ which leads to

$$j\omega L I\exp j\omega t+RI\exp j\omega t=V_m\exp j\omega t$$

The factor $\exp j\omega t$ is common and hence

$$(R+j\omega L)I=V_m$$

in which $R+j\omega L$ may be regarded as a *complex impedance*. The complex current I is now given by

$$I=\frac{V_m}{R+j\omega L}=\frac{V_m}{\sqrt{R^2+(\omega L)^2}}\exp-j\phi$$

with $\phi=\tan^{-1}(\omega L/R)$ and hence

$$i(t)=\mathrm{Re}\{I\exp j\omega t\}=\frac{V_m}{\sqrt{R^2+(\omega L)^2}}\cos(\omega t-\phi)$$

which, thankfully, is the same solution as before but arrived at with considerably greater ease.

Let us be clear about the approach. We have

(i) introduced a complex forcing function $V_m \exp j\omega t$ knowing that in reality the voltage source must be real *i.e.* $\text{Re}\{V_m \exp j\omega t\}$.

(ii) We solved the equations working with complex voltages and complex currents, $V \exp j\omega t$ and $I \exp j\omega t$ (or rather V and I since the time dependence $\exp j\omega t$ cancelled out).

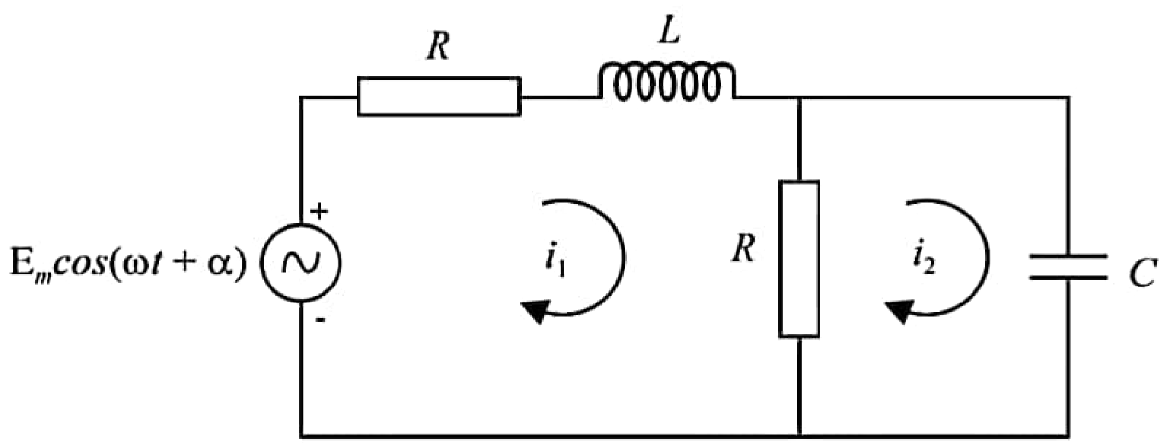
(iii) Since the actual voltage is given by $\text{Re}\{V_m \exp j\omega t\}$ the actual current is given by $\text{Re}\{I \exp j\omega t\} = \text{Re}\{|I| \exp j\psi \exp j\omega t\} = |I| \cos(\omega t + \psi)$.

(iv) Since the differential of $\exp(\mathrm{j}\omega t - \emptyset) = \exp(\mathrm{j}\omega t) . \exp(-\mathrm{j}\emptyset)$ is simply $j\omega . \exp(\mathrm{j}\omega t) . \exp(-\mathrm{j}\emptyset)$ and since we always take $\exp(\mathrm{j}\omega t)$ out as a common factor, you may see now that our differential equations turn into polynomial equations in jω (and you knew how to solve these at GCSE!)

This is a *very powerful* approach that will permit us to solve AC circuit problems very easily.

Example

Let's now do an example to show, formally, how we can solve AC problems. Let's imagine we want to find the steady state current, i_2, flowing through the capacitor in the following example



The two *KVL* loop equations may be written

$$Ri_1 + \frac{Ldi_1}{dt} + R(i_1 - i_2) = E_m \cos(\omega t + \alpha)$$

and

$$R(i_2 - i_1) + \frac{1}{C}\int i_2 \, dt = 0$$

Replacing $E_m \cos(\omega t + \alpha)$ by $E_m \exp j\ (\omega t + \alpha) = E_1 \exp j\,\omega t$ where $E_1 = E_m \exp j\alpha$ and further introducing I_1 and I_2 via $i_1 = I_1 \exp j\,\omega t$ and $i_2 = I_2 \exp j\,\omega t$ we obtain

$$(2R + j\omega L)I_1 - RI_2 = E_1$$

$$\left(R + \frac{1}{j\omega C}\right)I_2 - RI_1 = 0$$

and, after a little algebra

$$I_2 = \frac{E_1}{R + \frac{L}{CR} + j\left[\omega L - \frac{2}{\omega C}\right]} = \frac{E_m \exp j\alpha}{M + jN}$$

where $M = R + L/CR$ and $N = \omega L - 2/\omega C$. We note that this may be written, introducing $\tan\theta = N/M$

$$I_2 = \frac{E_m}{\sqrt{M^2 + N^2}} \exp j(\alpha - \theta)$$

and hence the actual current $i_2 = \mathrm{Re}(I_2 \exp j\omega t)$ may be written as

$$i_2(t) = \frac{E_m}{\sqrt{M^2 + N^2}} \cos(\omega t + \alpha - \theta)$$

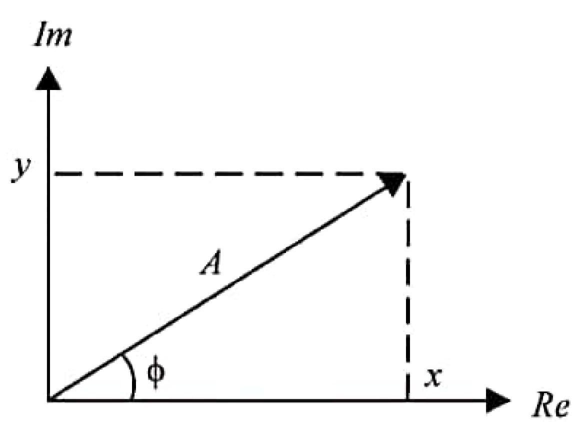
4. Phasors

We have just introduced a very powerful method of circuit analysis. In essence we have introduced the use of complex quantities to represent sinusoidal functions of time. The complex number $A \exp j\phi$ (often written $A\angle\phi$) when used in this context to represent $A\cos(\omega t + \phi)$ is called a *phasor*. Since the phase angle ϕ must be measured relative to some reference we may call the phasor $A\angle 0$ the *reference* phasor.

Since the phasor, $A \exp j\phi$, is complex it may be represented in Cartesian form $x + jy$ just like any other complex quantity and thus

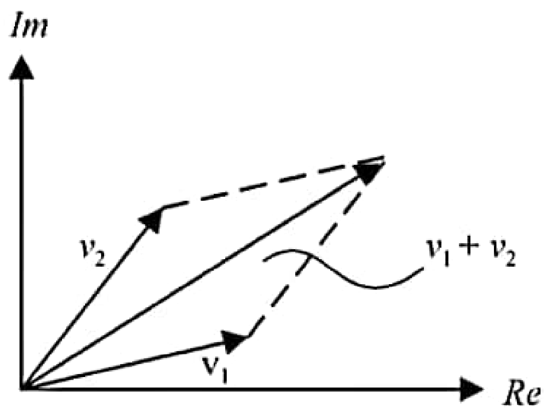
$$x = A\cos\phi$$
$$y = A\sin\phi$$
$$A = \sqrt{x^2 + y^2}$$

Further since the phasor is a complex quantity it is very easy to display it on an Argand diagram (also in this context called a phasor diagram). Thus the phasor $A \exp j\phi$ is drawn as a line of length A at an angle ϕ to the real axis.



We emphasise that this is a graphical representation of an actual sinusoid $A\cos(\omega t+\phi)$. The rules for addition, subtraction and multiplication of phasors are *identical* to those for complex numbers.

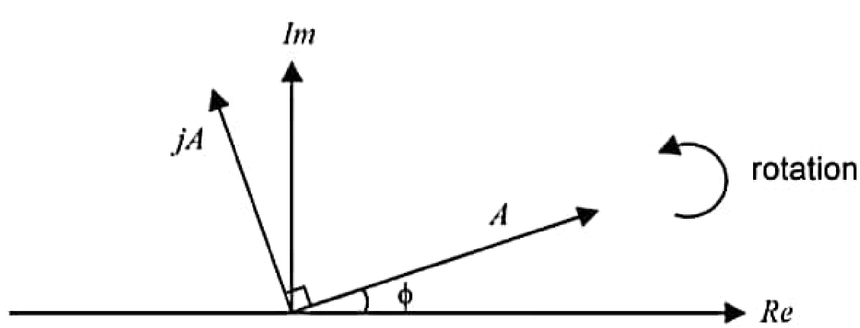
Thus **addition:**



For **multiplication** it is easiest to multiply the magnitudes and add the phases. Consider the effect of multiplying a phasor by j

$$j\,A\exp j\phi = \exp j\pi/2\;A\exp j\phi = A\exp j(\phi+\pi/2)$$

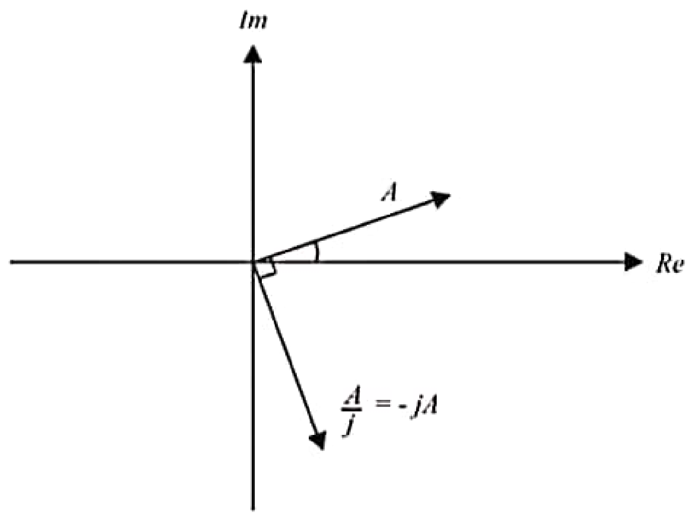
which causes the phasor to be rotated by 90°.



Similarly, dividing by j leads to

$$\frac{1}{j} A \exp j\phi = \exp(-j\pi/2)\, A \exp j\phi = A \exp j(\phi - \pi/2)$$

i.e. a rotation of –90°.



We finally note that it is usual to use rms values for the magnitude of phasors.

5. Phasor relations in passive elements

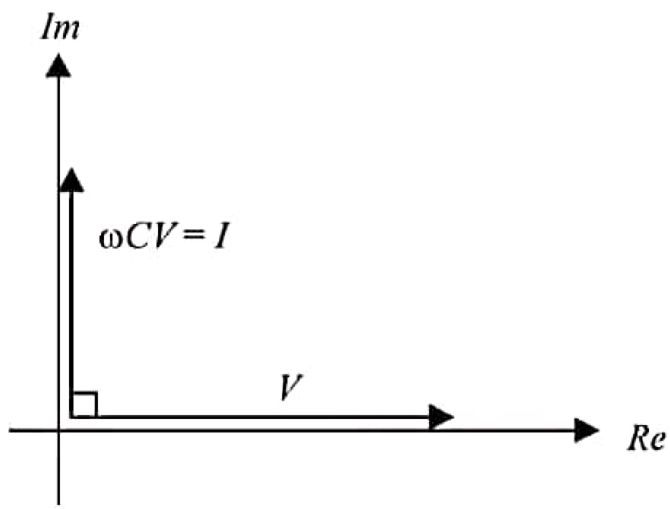
Consider now a voltage $V_m \cos \omega t$ applied to a **capacitor**. As we have indicated we elect to use the complex form $V_m \exp j\omega t$ and so omit the "real part" as we calculate the current via

$$i = C\frac{dv}{dt} = C\frac{d}{dt}\left(V_m \exp j\omega t\right) = j\omega C V_m \exp j\omega t$$

If we now drop the $\exp j\omega t$ notation and write the voltage phasor V_m as V and the current phasor as I we have

$$I = j\omega C V \quad \text{or} \quad V = \frac{1}{j\omega C} I$$

In terms of a phasor diagram, taking the voltage V as the reference



from which we confirm two things we already knew

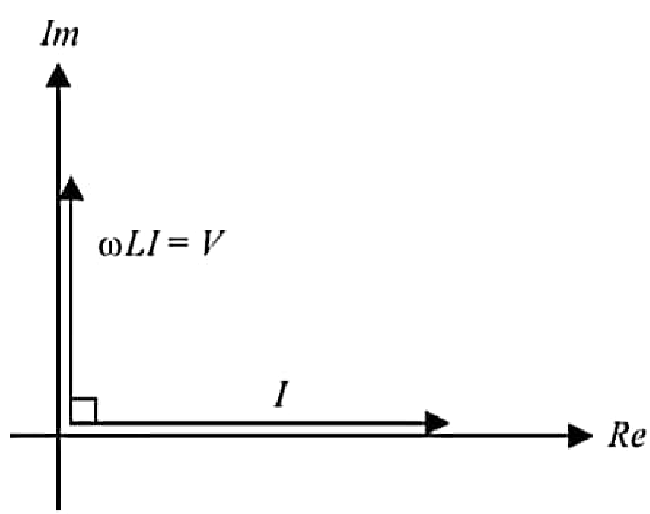
(i) the ratio of the voltage to the current is $\frac{1}{\omega C}$ - the **reactance**.

(ii) the current ***leads*** the voltage by 90°. The pre-multiplying factor j describes this.

For the **Inductance** an analogous procedure leads to

$$V = j\omega L I$$

Where the reactance is now $j\omega L$ and, if we now take, say, the current as the reference phasor we have

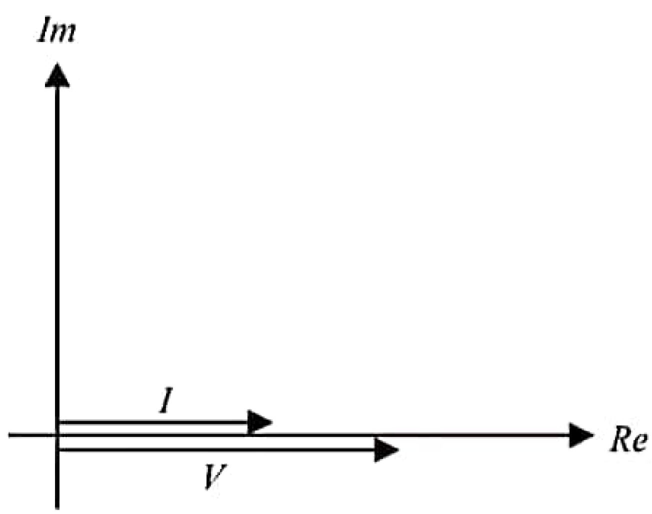


and here the current **lags** the voltage by 90°.

[It is important to get these relationships the right way around and as a check we may use the memory aid “CIVIL” – in a capacitor, the current leads the voltage **CIV**IL and in an inductor, the current lags the voltage CI**VIL**.]

Finally for a resistor we know that the current and voltage are in phase and hence, in phasor terms

$$V = IR$$



6. Phasors in circuit analysis

We are now in a position to summarise the method of analysis of AC circuits.

(i) We include all reactances as imaginary quantities $j\omega L\left(=jX_L\right)$ for an inductor and $1/j\omega C\left(=-jX_c\right)$ for a capacitor.

(ii) All voltages and currents are represented by phasors, which usually have rms magnitude, and one is chosen as a reference with zero phase angle.

(iii) All calculations are carried out in complex notation.

(iv) The magnitude and phase of, say, the current is obtained as $|I|\exp j\phi$. This can, if necessary, be converted back into a time varying expression $\sqrt{2}|I|\cos(\omega t+\phi)$.