UNIT - V TIME VARYING FIELDS

Faraday's Law of Electromagnetic Induction – It's Integral and Point Forms – Maxwell's Fourth Equation. Statically and Dynamically Induced E.M.F's – Simple Problems – Modified Maxwell's Equations for Time Varying Fields – Displacement Current.

Wave Equations – Uniform Plane Wave Motion in Free Space, Conductors and Dielectrics – Velocity, Wave Length, Intrinsic Impedence and Skin Depth – Poynting Theorem – Poynting Vector and its Significance**.**

Faraday's Law of electromagnetic Induction

Michael Faraday, in 1831 discovered experimentally that a current was induced in a conducting loop when the magnetic flux linking the loop changed. In terms of fields, we can say that a time varying magnetic field produces an electromotive force (emf) which causes a current in a closed circuit. The quantitative relation between the induced emf (the voltage that arises from conductors moving in a magnetic field or from changing magnetic fields) and the rate of change of flux linkage developed based on experimental observation is known as Faraday's law. Mathematically, the induced emf can be written as

Emf = $-\frac{d\phi}{dt}$ Volts (5.3)

where ϕ is the flux linkage over the closed path.

A non zero $\frac{d\phi}{dt}$ may result due to any of the following:

(a) time changing flux linkage a stationary closed path.

(b) relative motion between a steady flux a closed path.

(c) a combination of the above two cases.

The negative sign in equation (5.3) was introduced by Lenz in order to comply with the polarity of the induced emf. The negative sign implies that the induced emf will cause a current flow in the closed loop in such a direction so as to oppose the change in the linking magnetic flux which produces it. (It may be noted that as far as the induced emf is concerned, the closed path forming a loop does not necessarily have to be conductive).

If the closed path is in the form of N tightly wound turns of a coil, the change in the magnetic flux linking the coil induces an emf in each turn of the coil and total emf is the sum of the induced emfs of the individual turns, i.e., Emf = $-N\frac{d\phi}{dt}$ Volts (5.4)

By defining the total flux linkage as

$$\lambda = N\phi \quad (5.5)$$

The emf can be written as

Emf = $-\frac{d\lambda}{dt}$ (5.6)

Continuing with equation (5.3), over a closed contour 'C' we can write

Emf = $\oint_C \vec{E}.d\vec{l}$ (5.7)

where $\vec{E}$ is the induced electric field on the conductor to sustain the current.

Further, total flux enclosed by the contour 'C ' is given by

$$\phi = \int_S \vec{B}.d\vec{s} \quad (5.8)$$

Where S is the surface for which 'C' is the contour.

From (5.7) and using (5.8) in (5.3) we can write

$$\oint_C \vec{E}.d\vec{l} = -\frac{\partial}{\partial t}\oint_S \vec{B}.d\vec{s} \quad (5.9)$$

By applying stokes theorem

$$\int_S \nabla\times\vec{E}.d\vec{s} = -\int_S \frac{\partial\vec{B}}{\partial t}.d\vec{s} \quad (5.10)$$

Therefore, we can write

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad (5.11)$$

which is the Faraday's law in the point form

We have said that non zero $\frac{d\phi}{dt}$ can be produced in a several ways. One particular case is when a time varying flux linking a stationary closed path induces an emf. The emf induced in a stationary closed path by a time varying magnetic field is called a **transformer emf .**

Example: Ideal transformer

As shown in figure 5.1, a transformer consists of two or more numbers of coils coupled magnetically through a common core. Let us consider an ideal transformer whose winding has zero resistance, the core having infinite permittivity and magnetic losses are zero.

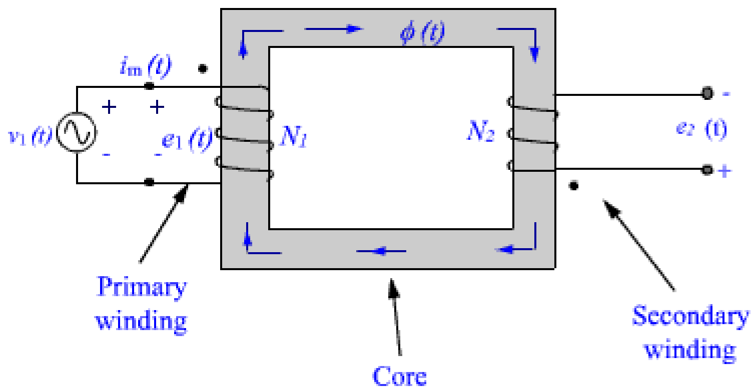

Fig 5.1: Transformer with secondary open

These assumptions ensure that the magnetization current under no load condition is vanishingly small and can be ignored. Further, all time varying flux produced by the primary winding will follow the magnetic path inside the core and link to the secondary coil without any leakage. If N_1 and N_2 are the number of turns in the primary and the secondary windings respectively, the induced emfs are

$$e_1 = N_1 \frac{d\phi}{dt} \qquad (5.12a)$$

$$e_2 = N_2 \frac{d\phi}{dt} \qquad (5.12b)$$

(The polarities are marked, hence negative sign is omitted. The induced emf is +ve at the dotted end of the winding.)

$$\therefore \frac{e_1}{e_2} = \frac{N_1}{N_2} \qquad (5.13)$$

i.e., the ratio of the induced emfs in primary and secondary is equal to the ratio of their turns. Under ideal condition, the induced emf in either winding is equal to their voltage rating.

$$\frac{v_1}{v_2} = \frac{N_1}{N_2} = a \qquad (5.14)$$

where 'a' is the transformation ratio. When the secondary winding is connected to a load, the current flows in the secondary, which produces a flux opposing the original flux. The net flux in the core decreases and induced emf will tend to decrease from the no load value. This causes the primary current to increase to nullify the decrease in the flux and induced emf. The current continues to increase till the flux in the core and the induced emfs are restored to the no load values. Thus the source supplies power to the primary winding and the secondary winding delivers the power to the load. Equating the powers

$$i_1 v_1 = i_2 v_2 \qquad (5.15)$$

$$\frac{i_2}{i_1} = \frac{v_1}{v_2} = \frac{e_1}{e_2} = \frac{N_1}{N_2} \qquad (5.16)$$

Further,

$$i_2 N_2 - i_1 N_1 = 0 \qquad (5.17)$$

i.e., the net magnetomotive force (mmf) needed to excite the transformer is zero under ideal condition.

Motional EMF:

Let us consider a conductor moving in a steady magnetic field as shown in the fig 5.2.

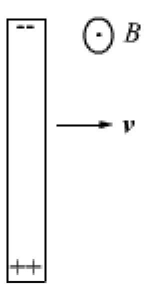

Fig 5.2

If a charge Q moves in a magnetic field $\vec{B}$, it experiences a force

$$\vec{F} = Q\vec{v} \times \vec{B} \qquad (5.18)$$

This force will cause the electrons in the conductor to drift towards one end and leave the other end positively charged, thus creating a field and charge separation continuous until electric and magnetic forces balance and an equilibrium is reached very quickly, the net force on the moving conductor is zero.

$\frac{\vec{F}}{Q} = \vec{v} \times \vec{B}$ can be interpreted as an induced electric field which is called the motional electric field

$\vec{E}_m = \vec{v} \times \vec{B}$ (5.19)

If the moving conductor is a part of the closed circuit C, the generated emf around the circuit is $\oint_c \vec{v} \times \vec{B} . d\vec{l}$. This emf is called the **motional emf**.

A classic example of **motional emf** is given in Additonal Solved Example No.1 .

Maxwell's Equation

Equation (5.1) and (5.2) gives the relationship among the field quantities in the static field. For time varying case, the relationship among the field vectors written as

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (5.20a)

$\nabla \times \vec{H} = \vec{J}$ (5.20b)

$\nabla . \vec{D} = \rho$ (5.20c)

$\nabla . \vec{B} = 0$ (5.20d)

In addition, from the principle of conservation of charges we get the equation of continuity

$$\nabla . \vec{J} = -\frac{\partial \rho}{\partial t} \quad (5.21)$$

The equation 5.20 (a) - (d) must be consistent with equation (5.21).

We observe that

$\nabla . \nabla \times \vec{H} = 0 = \nabla . \vec{J}$ (5.22)

Since $\nabla . \nabla \times \vec{A}$ is zero for any vector $\vec{A}$.

Thus $\nabla \times \vec{H} = \vec{J}$ applies only for the static case i.e., for the scenario when $\frac{\partial \rho}{\partial t} = 0$.
A classic example for this is given below .

Suppose we are in the process of charging up a capacitor as shown in fig 5.3.

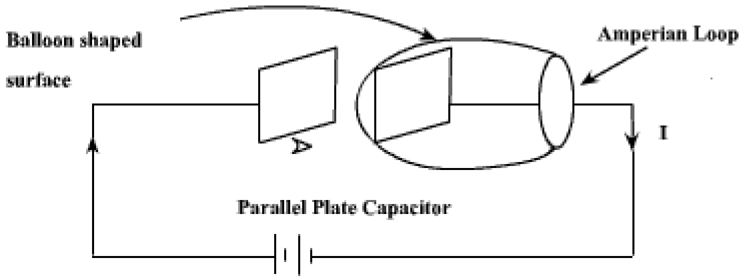

Fig 5.3

Let us apply the Ampere's Law for the Amperian loop shown in fig 5.3. $I_{enc} = I$ is the total current passing through the loop. But if we draw a baloon shaped surface as in fig 5.3, no current passes through this surface and hence $I_{enc} = 0$. But for non steady currents such as this one, the concept of current enclosed by a loop is ill-defined since it depends on what surface you use. In fact Ampere's Law should also hold true for time varying case as well, then comes the idea of displacement current which will be introduced in the next few slides.

We can write for time varying case,

$$\nabla.\left(\nabla\times\vec{H}\right) = 0 = \nabla.\vec{J} + \frac{\partial\rho}{\partial t}$$

$$= \nabla.\vec{J} + \frac{\partial}{\partial t}\nabla.\vec{D}$$

$$= \nabla.\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right) \qquad (5.23)$$

$$\therefore \nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \qquad (5.24)$$

The equation (5.24) is valid for static as well as for time varying case.

Equation (5.24) indicates that a time varying electric field will give rise to a magnetic field even in the absence of $\vec{J}$. The term $\frac{\partial\vec{D}}{\partial t}$ has a dimension of current densities $\left(A/m^2\right)$ and is called the displacement current density.

Introduction of $\frac{\partial\vec{D}}{\partial t}$ in $\nabla\times\vec{H}$ equation is one of the major contributions of Jame's Clerk Maxwell. The modified set of equations

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad (5.25a)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad (5.25b)$$

$$\nabla . \vec{D} = \rho \qquad (5.25c)$$

$$\nabla . \vec{B} = 0 \qquad (5.25d)$$

is known as the Maxwell's equation and this set of equations apply in the time varying scenario, static fields are being a particular case $\left(\frac{\partial}{\partial t} = 0 \right)$.

In the integral form

$$\oint_c \vec{E}.d\vec{l} = -\int_s \frac{\partial \vec{B}}{\partial t}.d\vec{S} \qquad (5.26a)$$

$$\oint_c \vec{H}.d\vec{l} = \int_s \left(J + \frac{\partial D}{\partial t} \right).d\vec{S} = I + \int_s \frac{\partial \vec{D}}{\partial t}.d\vec{S} \qquad (5.26b)$$

$$\int_V \nabla . \vec{D} dv = \oint_S \vec{D}.d\vec{S} = \int_V \rho dv \qquad (5.26c)$$

$$\oint \vec{B}.d\vec{S} = 0 \qquad (5.26d)$$

The modification of Ampere's law by Maxwell has led to the development of a unified electromagnetic field theory. By introducing the displacement current term, Maxwell could predict the propagation of EM waves. Existence of EM waves was later demonstrated by Hertz experimentally which led to the new era of radio communication.

10.2 General Wave Equations

In general the wave equations can be obtained by relating the space and time variations of the electric and magnetic fields, using the Maxwell's equations.

To obtain general wave equations, let us assume that the electric and magnetic fields exist in a linear, homogeneous and isotropic medium with the parameters μ_1 ε and σ. Also assume that the medium is source free which clearly gives the idea about the charge free medium. Assume that the medium obeys the ohm's law i.e. $\bar{J} = \sigma \bar{E}$. Then the Maxwell's equations are given by,

$$\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \qquad \ldots(1)$$

$$\nabla \times \bar{H} = \sigma \bar{E} + \varepsilon \frac{\partial \bar{E}}{\partial t} \qquad \ldots(2)$$

$$\nabla . \bar{B} = 0 \quad \text{i.e.} \quad \nabla . \bar{H} = 0 \qquad \ldots(3)$$

$$\nabla . \bar{D} = 0 \quad \text{i.e.} \quad \nabla . \bar{E} = 0 \qquad \ldots(4)$$

To eliminate $\bar{H}$ from equation (1), taking curl on both the sides of equation (1), we get,

$$\nabla \times (\nabla \times \bar{E}) = -\mu \left(\nabla \times \frac{\partial \bar{H}}{\partial t} \right) \qquad \ldots(5)$$

∇ operates indicates differentiation with respect to space while $\frac{\partial}{\partial t}$ operates differentiation with respect to time. Both are independent of each other, the operators can be interchanged.

So we get,

$$\nabla \times \nabla \times \bar{E} = -\mu \frac{\partial}{\partial t} (\nabla \times \bar{H}) \qquad \ldots(6)$$

Substituting value of $\nabla \times \bar{H}$ from equation (2), we get,

$$\nabla \times \nabla \times \bar{E} = -\mu \frac{\partial}{\partial t} \left[\sigma \bar{E} + \varepsilon \frac{\partial \bar{E}}{\partial t} \right]$$

$$\nabla \times \nabla \times \bar{E} = -\mu \sigma \frac{\partial \bar{E}}{\partial t} - \mu \varepsilon \frac{\partial^2 \bar{E}}{\partial t^2} \qquad \ldots(7)$$

Now according to the vector identity,

$$\nabla \times \nabla \times \bar{E} = \nabla (\nabla . \bar{E}) - \nabla^2 \bar{E} \qquad \ldots(8)$$

Substituting $\nabla . \bar{E} = 0$ from equation (4), we can modify equation (8) as,

$$\nabla \times \nabla \times \bar{E} = -\nabla^2 \bar{E} \qquad \ldots(9)$$

Substituting value of $\nabla \times \nabla \times \bar{E}$ from equation (9) in equation (2) we get,

$$-\nabla^2\overline{E} = -\mu\sigma\frac{\partial\overline{E}}{\partial t} - \mu\varepsilon\frac{\partial^2\overline{E}}{\partial t^2}$$

$$\therefore \quad \boxed{\nabla^2\overline{E} = \mu\sigma\frac{\partial\overline{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\overline{E}}{\partial t^2}} \quad \ldots(10)$$

This is the wave equation for the electric field $\overline{E}$. Now multiplying both the sides of equation (10) by ε,

$$\nabla^2(\varepsilon\overline{E}) = \mu\sigma\frac{\partial\varepsilon\overline{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\varepsilon\overline{E}}{\partial t^2}$$

$$\text{i.e} \quad \boxed{\nabla^2\overline{D} = \mu\sigma\frac{\partial\overline{D}}{\partial t} + \mu\varepsilon\frac{\partial^2\overline{D}}{\partial t^2}} \quad \ldots(11)$$

This is the wave equation for $\overline{D}$ in uniform medium.

Exactly on the similar lines, the wave equation for $\overline{H}$ can be obtained by taking curl on both the sides of equation (2), we get,

$$\nabla\times(\nabla\times\overline{H}) = \nabla\times(\sigma\overline{E}) + \varepsilon\nabla\times\frac{\partial\overline{E}}{\partial t} \quad \ldots(12)$$

As ∇ operator and $\frac{\partial}{\partial t}$ represent independent relationship between the two, we can interchange them as follows.

$$\nabla\times\nabla\times\overline{H} = \sigma(\nabla\times\overline{E}) + \varepsilon\frac{\partial}{\partial t}(\nabla\times\overline{E}) \quad \ldots(13)$$

Substituting $\nabla\times\overline{E} = -\mu\frac{\partial\overline{H}}{\partial t}$ in equation (12), we get,

$$\nabla\times\nabla\times\overline{H} = \sigma\left(-\mu\frac{\partial\overline{H}}{\partial t}\right) + \varepsilon\left(-\mu\frac{\partial\overline{H}}{\partial t}\right)$$

$$\therefore \quad \nabla\times\nabla\times\overline{H} = -\mu\sigma\frac{\partial\overline{H}}{\partial t} - \mu\varepsilon\frac{\partial\overline{H}}{\partial t} \quad \ldots(14)$$

From the vector indentity,

$$\nabla\times\nabla\times\overline{H} = \nabla(\nabla.\overline{H}) - \nabla^2\overline{H} \quad \ldots(15)$$

Substituting $\nabla.\overline{H} = 0$ from equation (4) in equation (15), we get

$$\nabla\times\nabla\times\overline{H} = -\nabla^2\overline{H} \quad \ldots(16)$$

Substituting value of $\nabla\times\nabla\times\overline{H}$ in equation (14) we get,

$$-\nabla^2\overline{H} = -\mu\sigma\frac{\partial\overline{H}}{\partial t} - \mu\varepsilon\frac{\partial\overline{H}}{\partial t}$$

$$\text{i.e.} \quad \boxed{\nabla^2\overline{H} = \mu\sigma\frac{\partial\overline{H}}{\partial t} + \mu\varepsilon\frac{\partial^2\overline{H}}{\partial t}} \quad \ldots(17)$$

This is the wave equation for the magnetic field $\overline{H}$. Now multiplying both the sides by μ, we get,

$$\nabla^2(\mu\overline{H}) = \mu\sigma\frac{\partial\mu\overline{H}}{\partial t} + \mu\varepsilon\frac{\partial^2(\mu\overline{H})}{\partial t}$$

$$\therefore \quad \boxed{\nabla^2 \overline{B} = \mu \sigma \frac{\partial \overline{B}}{\partial t} + \mu \varepsilon \frac{\partial^2 \overline{B}}{\partial t}} \quad \ldots(18)$$

This is the wave equation for $\overline{D}$ in the uniform medium. Hence in general we can write,

$$\nabla^2 \begin{bmatrix} \overline{E} \\ \overline{D} \\ \overline{H} \\ \overline{B} \end{bmatrix} = \mu \sigma \frac{\partial}{\partial t} \begin{bmatrix} \overline{E} \\ \overline{D} \\ \overline{H} \\ \overline{B} \end{bmatrix} + \mu \varepsilon \frac{\partial^3}{\partial t^2} \begin{bmatrix} \overline{E} \\ \overline{D} \\ \overline{H} \\ \overline{B} \end{bmatrix} \quad \ldots(19)$$

Above equation is three dimensional equation for all the field vectors.

9.2 Uniform Plane Wave in Free Space

Assume an electromagnetic wave travelling in free space. Consider that an electric field is in x-direction; while a magnetic field is in y-direction. Both the fields will not vary with x and y; but with z only. They will also change with time as the wave propagates in free space.

Consider Maxwell's equation expressed in $\overline{E}$ and $\overline{H}$ as

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t}$$

Let us assume that a free space is perfect dielectric, then $\overline{J} = 0$,

$$\therefore \quad \nabla \times \overline{H} = \frac{\partial \overline{D}}{\partial t}$$

Expressing $\overline{D}$ in rectangular co-ordinate system,

$$\nabla \times \overline{H} = \frac{\partial}{\partial t}\left[D_x \bar{a}_x + D_y \bar{a}_y + D_z \bar{a}_z\right]$$

Writing curl of $\overline{H}$ on left of equation,

$$\left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right]\bar{a}_x + \left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right]\bar{a}_y + \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right]\bar{a}_z$$

$$= \frac{\partial}{\partial t}\left[D_x \bar{a}_x + D_y \bar{a}_y + D_z \bar{a}_z\right]$$

As $\overline{H}$ is in y-direction, $H_x = H_z = 0$,

$$\therefore\ -\frac{\partial H_y}{\partial z}\bar{a}_x + \frac{\partial H_y}{\partial x}\bar{a}_z = \frac{\partial}{\partial t}\left[D_x \bar{a}_x + D_y \bar{a}_y + D_z \bar{a}_z\right]$$

Also H_y is not changing with x, as it is uniform in x-y plane, so $\frac{\partial H_y}{\partial x} = 0$

$$\therefore \qquad -\frac{\partial H_y}{\partial z}\bar{a}_x = \frac{\partial}{\partial t}\left[D_x \bar{a}_x + D_y \bar{a}_y + D_z \bar{a}_z\right]$$

Equating L.H.S. and R.H.S. of above equation directionwise, we can write,

$$-\frac{\partial H_y}{\partial z} = \frac{\partial D_x}{\partial t}$$

$$\therefore \qquad -\frac{\partial H_y}{\partial z} = \varepsilon\frac{\partial E_x}{\partial t} \quad .... \because \ \overline{D} = \varepsilon\overline{E}$$

$$\therefore \qquad \frac{\partial H_y}{\partial z} = -\varepsilon\frac{\partial E_x}{\partial t} \qquad ...(1)$$

Now consider Maxwell's equation derived from Faraday's law,

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$$

Using rectangular co-ordinate system, we can write,

$$\left[\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}\right]\bar{a}_x + \left[\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right]\bar{a}_y + \left[\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right]\bar{a}_z$$

$$= -\frac{\partial}{\partial t}\left[B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z\right]$$

As $\overline{\mathbf{E}}$ is in x-direction, $E_y = E_z = 0$,

$$\therefore \quad \frac{\partial E_x}{\partial z}\,\bar{\mathbf{a}}_y + \frac{\partial E_x}{\partial y}\,\bar{\mathbf{a}}_z = -\frac{\partial}{\partial t}\left[B_x\,\bar{\mathbf{a}}_x + B_y\,\bar{\mathbf{a}}_y + B_z\,\bar{\mathbf{a}}_z\right]$$

Also component E_x is not varying with y, as it is uniform in x-y plane, so $\frac{\partial E_x}{\partial y} = 0$,

$$\therefore \quad \frac{\partial E_x}{\partial z}\,\bar{\mathbf{a}}_y = -\frac{\partial}{\partial t}\left[B_x\,\bar{\mathbf{a}}_x + B_y\,\bar{\mathbf{a}}_y + B_z\,\bar{\mathbf{a}}_z\right]$$

Equating L.H.S. and R.H.S. of above equation directionwise, we can write,

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$

$$\therefore \quad \frac{\partial E_x}{\partial z} = -\mu\,\frac{\partial H_y}{\partial t} \quad \text{.........} \quad \overline{\mathbf{B}} = \mu\,\overline{\mathbf{H}}$$

$$\therefore \quad \frac{\partial H_y}{\partial t} = -\frac{1}{\mu}\,\frac{\partial E_x}{\partial z} \qquad \text{... (2)}$$

Differentiating equation (1) with respect to t,

$$\frac{\partial}{\partial t}\left[\frac{\partial H_y}{\partial z}\right] = -\varepsilon\,\frac{\partial^2 E_x}{\partial t^2} \qquad \text{... (3)}$$

Now differentiating equation (2) with respect to z,

$$\frac{\partial}{\partial z}\left[\frac{\partial H_y}{\partial t}\right] = -\frac{1}{\mu}\,\frac{\partial^2 E_x}{\partial z^2} \qquad \text{... (4)}$$

Now observe L.H.S. of equations (3) and (4). As we can change order of differentiation, L.H.S. of equations (3) and (4) is same. So equating R.H.S. of both the equations,

$$\therefore \quad \varepsilon\,\frac{\partial^2 E_x}{\partial t^2} = \frac{1}{\mu}\,\frac{\partial^2 E_x}{\partial z^2}$$

$$\therefore \quad \frac{\partial^2 E_x}{\partial t^2} = \frac{1}{\mu\varepsilon}\,\frac{\partial^2 E_x}{\partial z^2} \qquad \text{... (5)}$$

According to the results in physics,

$$v = \frac{1}{\sqrt{\mu\varepsilon}}$$

where v is the velocity of propagation also called wave **velocity**. For the free space it is denoted by c and its value is 3×10^8 m/s.

Hence we can rewrite equation (5) as,

$$\frac{\partial^2 E_x}{\partial t^2} = v^2 \frac{\partial^2 E_x}{\partial z^2} \quad \text{... (6)}$$

Above equation is the **wave equation** and it is differential equation of second order. Solving this equation mathematically, the solution is given by

$$E_x = E_m^+ \cos(\omega t - \beta z) + E_m^- \cos(\omega t + \beta z) \text{ V / m} \quad \text{... (7)}$$

Above equation (7) is a sinusoidal function consisting two components of an electric field ; one in forward direction and other in backward direction. The wave consists one component of the field travelling in positive z direction having amplitude E_m^+ ; while other component having amplitude E_m^- travelling in negative z direction.

We can rewrite equation (7) as follows,

$$E_x = E_m^+ \cos\omega\left(t - \frac{\beta}{\omega} z\right) + E_m^- \cos\omega\left(t + \frac{\beta}{\omega} z\right) \text{ V / m} \quad \text{... (8)}$$

Two partial differentiations of equation (8) with respect to z and t yields a similar equation given by

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{\beta^2}{\omega^2}\left(\frac{\partial^2 E_x}{\partial t^2}\right) \quad \text{... (9)}$$

It is clear that equations (6) and (9) are similar equations. So comparing these two equations we can get another expression for velocity as,

$$v = \frac{\omega}{\beta} \text{ m/s} \quad \text{... (10)}$$

We can obtain similar type of equations for magnetic field $\overline{\mathbf{H}}$ by considering equation (2) and putting value of E_x from equation (8),

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu}\frac{\partial}{\partial z}\left[E_m^+ \cos\omega\left(t - \frac{\beta}{\omega} z\right) + E_m^- \cos\omega\left(t + \frac{\beta}{\omega} z\right)\right]$$

$$\therefore \quad \frac{\partial H_y}{\partial t} = -\frac{1}{\mu}\left[\beta E_m^+ \sin\omega\left(t - \frac{\beta}{\omega} z\right) - \beta E_m^- \sin\omega\left(t + \frac{\beta}{\omega} z\right)\right]$$

Integrating both sides with respect to time, we get,

$$H_y = \frac{\beta}{\omega\mu} E_m^+ \cos\omega\left(t - \frac{\beta}{\omega} z\right) - \frac{\beta}{\omega\mu} E_m^- \cos\omega\left(t + \frac{\beta}{\omega} z\right)$$

$$\therefore \quad H_y = H_m^+ \cos(\omega t - \beta z) - H_m^- \cos(\omega t + \beta z) \text{ A / m} \quad \text{... (11)}$$

This equation is similar to equation (7) representing two components of a magnetic field one in forward direction while other in backward direction.

From equations (7) and (11) it is clear that when we assume x component for $\bar{E}$, it results in y component for $\bar{H}$. Both $\bar{E}$ and $\bar{H}$ are in time phase and both are perpendicular to each other. Both these fields lie in a plane which is perpendicular to the direction of wave propagation. Thus $\bar{E}$ and $\bar{H}$ together form **transverse electromagnetic (TEM)** wave ; with one forward travelling wave in the positive z-direction with velocity $\frac{\omega}{\beta}$ and another backward travelling wave in negative z-direction with the same velocity. Thus $\bar{E}$ and $\bar{H}$ are only the functions of direction of travel and time.

In general, when any wave propagates in the medium, it gets attenuated. The amplitude of the signal reduces. This is represented by an **attenuation constant** α. It is measured in neper per meter (Np/m). But practically it is expressed in decibel (dB). The conversion between a basic unit neper (Np) and decibel (dB) is given by

$$1 \text{ Np} = 8.686 \text{ dB}$$

It is also observed that when a wave propagates, phase change also takes place. Such a phase change is expressed by **a phase constant** β. It is measured in radian per meter (rad/m).

So attenuation constant (α) and phase constant (β) together constitutes a propagation constant of medium for uniform plane wave. It is represent by γ. It is expressed per unit length as

$$\gamma = \alpha + j\beta \qquad \text{... (12)}$$

The ratio of amplitudes of $\bar{E}$ to $\bar{H}$ of the waves in either direction is called **intrinsic impedance** of the material in which wave is travelling. It is denoted by η and given by,

$$\eta = \frac{E_m^+}{H_m^+} = -\frac{E_m^-}{H_m^-} = \frac{\omega\mu}{\beta} = v \cdot \mu \ \Omega \qquad \text{... (13)}$$

But as we know, $v = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}$,

$$\therefore \qquad \eta = \frac{\mu}{\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} \ \Omega \qquad \text{... (14)}$$

For free space, intrinsic impedance is denoted by η_0,

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 120\,\pi\,\Omega = 377\,\Omega \quad \text{... (15)}$$

and

$$v = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \approx 3 \times 10^8 \;\; \text{m/s} \quad \text{... (16)}$$

In general, wave repeats itself after 2π radians. In otherwords, if λ is the length of one cycle of sinusoidal signal, then signal changes phase by 360° or 2π radians. So we can write relation between λ and β as,

$$\boxed{\lambda = \frac{2\pi}{\beta} \;\; \text{m}} \quad \text{... (17)}$$

Multiplying both sides of equation (17) by frequency f,

$$\therefore \quad (f)\,(\lambda) = \frac{2\pi f}{\beta} = \frac{\omega}{\beta} = v$$

Thus velocity of propagation or wave velocity is given by,

$$\boxed{v = f\,\lambda \;\; \text{m/s}} \quad \text{... (18)}$$

9.3 Wave Equations in Phasor Form

An electromagnetic wave in a medium can be completely defined if intrinsic impedance (η) and propagation constant (γ) of a medium is known. Thus it is necessary to derive the expressions for η and γ in terms of the properties of a medium such as permeability (μ), permittivity (ε), conductivity (σ) etc.

Consider Maxwell's equation derived from Faraday's law,

$$\nabla \times \overline{\mathbf{E}} = -\frac{\partial \overline{\mathbf{B}}}{\partial t} = -\mu \frac{\partial \overline{\mathbf{H}}}{\partial t} \quad \text{... (1)}$$

Taking curl on both the sides of the equation,

$$\therefore \quad \nabla \times \nabla \times \overline{\mathbf{E}} = -\mu \left[\nabla \times \frac{\partial \overline{\mathbf{H}}}{\partial t} \right]$$

$$\therefore \quad \nabla \times \nabla \times \overline{\mathbf{E}} = -\mu \left[\frac{\partial}{\partial t} \left(\nabla \times \overline{\mathbf{H}} \right) \right] \quad \text{... (2)}$$

Using vector identity to the left of equation (2),

$$\therefore \quad \nabla(\nabla \cdot \overline{\mathbf{E}}) - \nabla^2 \overline{\mathbf{E}} = -\mu \left[\frac{\partial}{\partial t}(\nabla \times \overline{\mathbf{H}})\right] \quad \text{... (3)}$$

But according to another Maxwell's equation,

$$\nabla \times \overline{\mathbf{H}} = \bar{\mathbf{J}} + \frac{\partial \overline{\mathbf{D}}}{\partial t}$$

Putting value of $\nabla \times \overline{\mathbf{H}}$ in equation (3),

$$\therefore \quad \nabla(\nabla \cdot \overline{\mathbf{E}}) - \nabla^2 \overline{\mathbf{E}} = -\mu \left[\frac{\partial}{\partial t}\left(\bar{\mathbf{J}} + \frac{\partial \overline{\mathbf{D}}}{\partial t}\right)\right] \quad \text{... (4)}$$

Since most of the regions are source or charge free,

$$\therefore \quad \nabla \cdot \overline{\mathbf{E}} = 0$$

$$\therefore \quad \nabla(\nabla \cdot \overline{\mathbf{E}}) = 0$$

Putting value of $\nabla(\nabla \cdot \overline{\mathbf{E}})$ in equation (4), assuming charge free medium,

$$-\nabla^2 \overline{\mathbf{E}} = -\mu \left[\frac{\partial}{\partial t}\left(\bar{\mathbf{J}} + \frac{\partial \overline{\mathbf{D}}}{\partial t}\right)\right]$$

Making both sides positive,

$$\nabla^2 \overline{\mathbf{E}} = \mu \left[\frac{\partial}{\partial t}\left(\bar{\mathbf{J}} + \frac{\partial \overline{\mathbf{D}}}{\partial t}\right)\right] \quad \text{... (5)}$$

Consider a general electromagnetic wave with both the fields, $\overline{\mathbf{E}}$ and $\overline{\mathbf{H}}$ varying with respect to time. When any field varies with respect to time, its partial derivative taken with respect to time can be replaced by $j\omega$. Rewriting equation (5) in phasor form,

$$\therefore \quad \nabla^2 \overline{\mathbf{E}} = \mu \left[j\omega \left(\bar{\mathbf{J}} + j\omega \overline{\mathbf{D}}\right)\right]$$

$$\therefore \quad \nabla^2 \overline{\mathbf{E}} = j\omega\mu \left[(\sigma \overline{\mathbf{E}}) + j\omega(\varepsilon \overline{\mathbf{E}})\right]$$

$$\therefore \quad \nabla^2 \overline{\mathbf{E}} = \left[j\omega\sigma\mu \overline{\mathbf{E}} + (j\omega)\varepsilon\mu \overline{\mathbf{E}}\right]$$

$$\therefore \quad \nabla^2 \overline{\mathbf{E}} = \left[j\omega\mu(\sigma + j\omega\varepsilon)\right]\overline{\mathbf{E}} \quad \text{... (6)}$$

In similar way, we can write another phasor equation as,

$$\nabla^2 \overline{H} = [j\omega\mu(\sigma + j\omega\varepsilon)]\overline{H} \qquad \text{... (7)}$$

The terms inside the bracket in equations (6) and (7) are exactly similar and represent the properties of the medium in which wave is propagating. The total bracket is the square of a propagation constant γ, hence we can rewrite equations (6) and (7) as,

$$\nabla^2 \overline{E} = \gamma^2 \overline{E} \quad \text{and}$$

$$\nabla^2 \overline{H} = \gamma^2 \overline{H}$$

So the propagation constant γ can be expressed in terms of properties of the medium as,

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} \qquad \text{... (8)}$$

The real and imaginary parts of γ are attenuation constant (α) and phase constant (β) and both can be expressed in terms of the properties of the medium,

$$\therefore \quad \alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right)} \qquad \text{... (9)}$$

$$\text{and} \quad \beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right)} \qquad \text{... (10)}$$

The intrinsic impedance of a medium can be expressed interms of the properties of a medium and is given by,

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \qquad \text{... (11)}$$

It can also be expressed in polar form as $|\eta| \angle\theta$ where

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}} \quad \text{and}$$

$$\tan 2\theta = \frac{\sigma}{\omega\varepsilon} \qquad 0° < \theta < 45°$$

Let us summarize the equations which are helpful in describing the electromagnetic waves (uniform plane waves). Table 9.1 lists equations describing the propagation of EM waves in a medium.