6.3 Plane wave reflection from media interface at oblique incidence

- We will consider the problem of a plane wave
  - obliquely incident on a plane interface
  - between two lossy conducting regions
- We will first consider two particular cases of this problem as follows:
  - the electric field is in the xz plane (parallel polarization)
  - the electric field is in normal to the xz plane (perpendicular polarization)

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- Any arbitrary incident plane wave can be expressed
  - as a linear combination of these two principal polarizations
- The plane of incidence is that plane containing
  - the normal vector to the interface and
  - the direction of propagation vector of the incident wave

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- For Fig. 6.4, this is the xz plane
- For perpendicular polarization (TE),
  - electric field is perpendicular to the plane of incidence
- For parallel polarization (TM),
  - electric field is parallel to the plane of incidence

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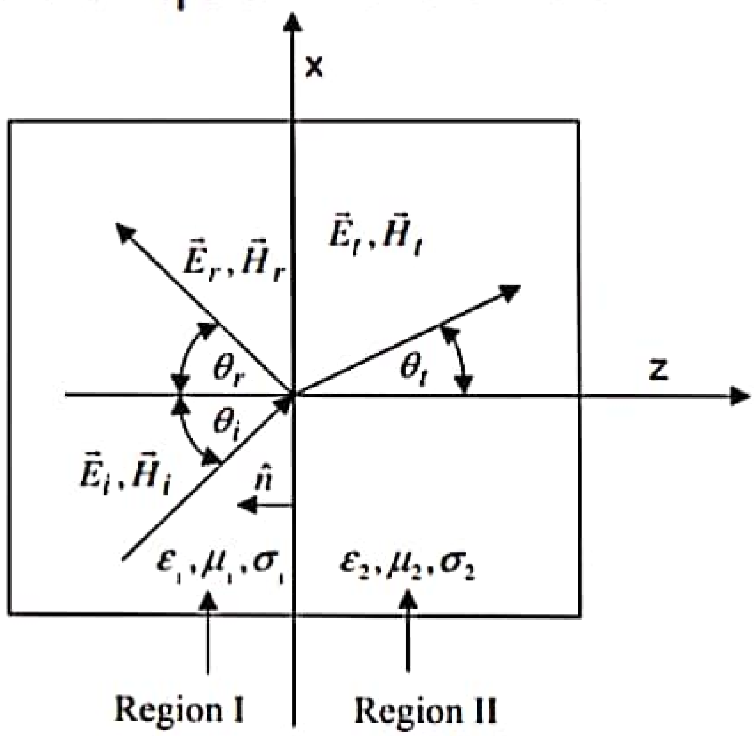

Fig. 6.5 Oblique incidence of plane EM wave at a media interface

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6.3.1 Perpendicular polarization (TE):

- In this case, electric field vector is perpendicular to the xz plane,
- Hence, it will have component along the y-axis
- Since the electric field is transversal to the plane of incidence
- They are also known transverse electric (TE) waves

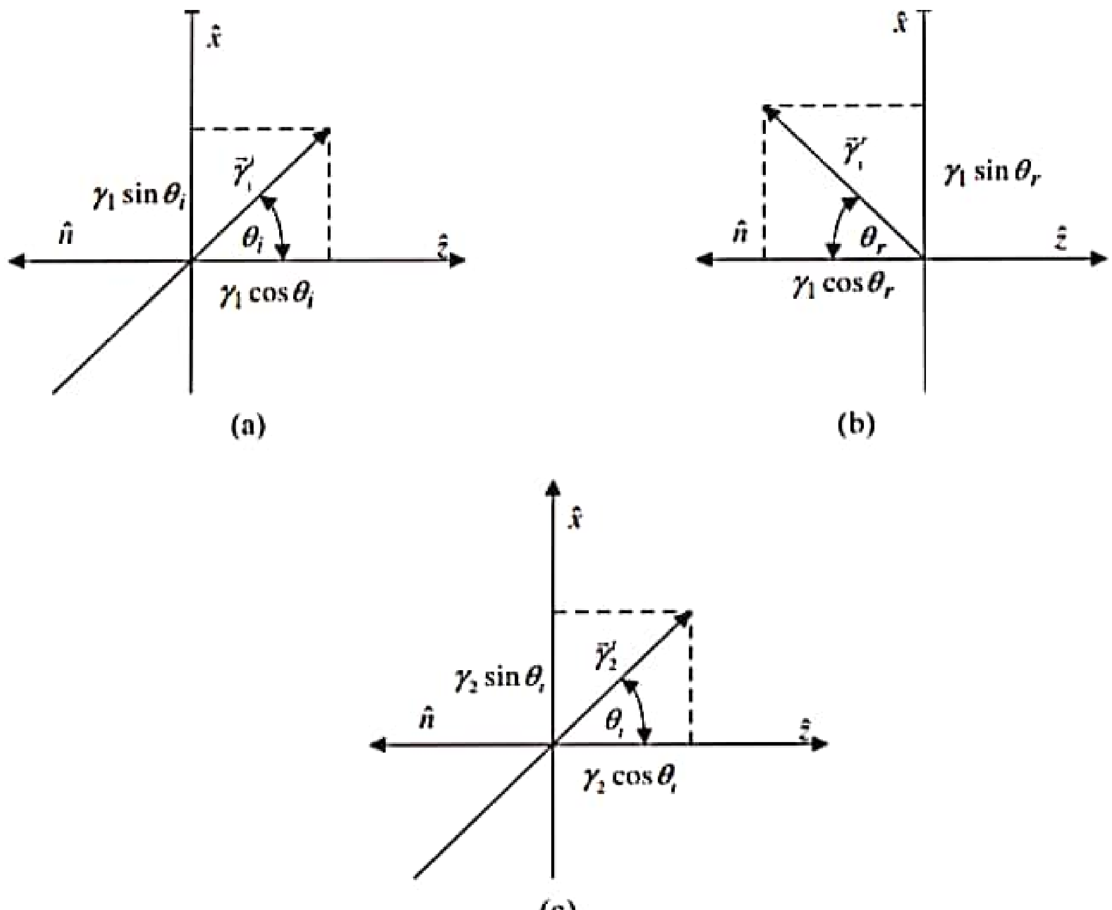

Fig. 6.6 Wave propagation vector for (a) incident (b) reflected and (c) transmitted EM waves at oblique incidence

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- Let us assume that the incident wave propagates in the first quadrant of xz plane without loss of generality and
- $\vec{\gamma}_1{}^i$ (incident propagation vector) makes an angle θ_i with the normal (see Fig. 6.6 (a))

$$\vec{\gamma}_1^i \bullet \vec{z}' = (\gamma_1 \cos\theta_i \hat{z} + \gamma_1 \sin\theta_i \hat{x}) \bullet (z\hat{z} + x\hat{x}) = \gamma_1 \cos\theta_i z + \gamma_1 \sin\theta_i x = \gamma_1 (z\cos\theta_i + x\sin\theta_i)$$

$$\vec{E}_i = E_0 e^{-\gamma_1 (z\cos\theta_i + x\sin\theta_i)} \hat{y}$$

$$\because \nabla \times \vec{E}_i = -j\omega\mu_1 \vec{H}_i \Rightarrow \vec{H}_i = \frac{\nabla \times \vec{E}_i}{-j\omega\mu_1}$$

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$$= \frac{1}{-j\omega\mu_1}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_0 e^{-\gamma_1(z\cos\theta_i + x\sin\theta_i)} & 0 \end{vmatrix} = \frac{E_0}{-j\omega\mu_1}\left\{\left(-\frac{\partial e^{-\gamma_1(z\cos\theta_i + x\sin\theta_i)}}{\partial z}\hat{x}\right) + \left(\frac{\partial e^{-\gamma_1(z\cos\theta_i + x\sin\theta_i)}}{\partial x}\hat{z}\right)\right\}$$

$$= \frac{E_0}{j\omega\mu_1}\left\{\left(\frac{\partial e^{-\gamma_1(z\cos\theta_i + x\sin\theta_i)}}{\partial z}\hat{x}\right) - \left(\frac{\partial e^{-\gamma_1(z\cos\theta_i + x\sin\theta_i)}}{\partial x}\hat{z}\right)\right\} = \frac{E_0\gamma_1}{j\omega\mu_1} e^{-\gamma_1(z\cos\theta_i + x\sin\theta_i)}\{-\cos\theta_i\hat{x} + \hat{z}\sin\theta_i\}$$

$$= \frac{E_0}{\eta_1} e^{-\gamma_1(z\cos\theta_i + x\sin\theta_i)}(-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)$$

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- Let us assume that the reflected wave propagates in the second quadrant of xz plane and
- $\vec{\gamma}_1^{\,r}$ (reflected propagation vector) makes an angle θ_r with the normal (see Fig. 6.6 (b))

$$\vec{\gamma}_1^{\,r} \bullet \vec{z}\,' = \left(-\gamma_1 \cos\theta_r \hat{z} + \gamma_1 \sin\theta_r \hat{x}\right) \bullet \left(z\hat{z} + x\hat{x}\right) = -\gamma_1 \cos\theta_r z + \gamma_1 \sin\theta_r x \equiv \gamma_1\left(-z\cos\theta_r + x\sin\theta_r\right)$$

$$\vec{E}_r = E_0 \Gamma_{TE} e^{-\gamma_1\left(-z\cos\theta_r + x\sin\theta_r\right)}\,\hat{y}$$

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- Note that $\bar{\gamma}_1^r$ and $\bar{\gamma}_1^i$ will have the same magnitude
  - since both the waves are still in the same region I,
  - only their direction changes
- Since the Poynting vector must be negative like the previous case of normal incidence,

$$\vec{H}_r = \frac{E_0}{\eta_1}\Gamma_{TE} e^{-\gamma_1(-z\cos\theta_r + x\sin\theta_r)}\left(\hat{x}\cos\theta_r + \hat{z}\sin\theta_r\right)$$

- You could also use the Maxwell's curl equation below to find this

$$\vec{H}_r = \frac{\nabla \times \vec{E}_r}{-j\omega\mu_1}$$

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- The transmitted fields will have similar expression with the incident fields except
  - that now the θ_i should be replaced by θ_t (angle that transmitted propagation vector makes with the normal),
  - γ_1 should be replaced by γ_2 (wave is in region II now) and
  - multiplication by (transmission coefficient)
- The transmitted fields are $\vec{E}_t = \hat{y}E_0\tau_{TE}e^{-\gamma_2(z\cos\theta_t + x\sin\theta_t)}$

$$\vec{H}_t = \frac{\nabla\times\vec{E}_t}{-j\omega\mu_2} = \frac{E_0\tau_{TE}}{\eta_2}e^{-\gamma_2(z\cos\theta_t + x\sin\theta_t)}(-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)$$

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Table 6.5 Fields in two regions (oblique incidence: perpendicular polarization)

Region I (lossy medium 1)	Region II (lossy medium 2)
$\vec{E}_i = E_0 e^{-\gamma_1(z\cos\theta_i + x\sin\theta_i)}\hat{y}$ $\vec{H}_i = \frac{E_0}{\eta_1} e^{-\gamma_1(z\cos\theta_i + x\sin\theta_i)}(-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)$ $\vec{E}_r = E_0\Gamma_{TE} e^{-\gamma_1(-z\cos\theta_r + x\sin\theta_r)}\hat{y}$ $\vec{H}_r = \frac{E_0\Gamma_{TE}}{\eta_1} e^{-\gamma_1(z\cos\theta_r + x\sin\theta_r)}(\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)$	$\vec{E}_t = \hat{y}E_0\tau_{TE} e^{-\gamma_2(z\cos\theta_t + x\sin\theta_t)}$ $\vec{H}_t = \frac{E_0\tau_{TE}}{\eta_2} e^{-\gamma_2(z\cos\theta_t + x\sin\theta_t)}(-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)$

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- Equating the tangential components of electric field
  - (electric field has only E_y component and it is tangential at the interface z=0) and
- magnetic field
  - (magnetic field has two components: H_x and H_z and only H_x is tangential at the interface z=0)
- at z=0 gives $$e^{-\gamma_1 x \sin\theta_i} + \Gamma_{TE} e^{-\gamma_1 x \sin\theta_r} = \tau_{TE} e^{-\gamma_2 x \sin\theta_t}$$

$$\frac{-1}{\eta_1}\cos\theta_i e^{-\gamma_1 x \sin\theta_i} + \frac{\Gamma_{TE}}{\eta_1}\cos\theta_r e^{-\gamma_1 x \sin\theta_r} = -\frac{\tau_{TE}}{\eta_2}\cos\theta_t e^{-\gamma_2 x \sin\theta_t}$$

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- If E_x and H_y are to be continuous at the interface $z = 0$ for all x,
- then, this x variation must be the same on both sides of the equations (also known as *phase matching condition*)

$$\gamma_1 \sin\theta_i = \gamma_1 \sin\theta_r = \gamma_2 \sin\theta_t$$

$$\Rightarrow \theta_i = \theta_r; \gamma_1 \sin\theta_i = \gamma_2 \sin\theta_t$$

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- The first is Snell's law of reflection
  - which states that the angle of incidence equals the angle of reflection
- The second result is the Snell's law of refraction
  - (refraction is the change in direction of a wave due to change in velocity from one medium to another medium)
- Also note that refractive index of a medium is defined as

$$n = \frac{c}{v_p} = \frac{\sqrt{\mu_r \varepsilon_r \mu_0 \varepsilon_0}}{\sqrt{\mu_0 \varepsilon_0}} = \sqrt{\mu_r \varepsilon_r}$$

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- hence, for a lossless dielectric media,

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{\gamma_2}{\gamma_1} = \frac{\beta_2}{\beta_1} = \frac{\sqrt{\mu_2\varepsilon_2}}{\sqrt{\mu_1\varepsilon_1}} = \frac{v_1}{v_2} = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}} = \frac{n_2}{n_1}$$

- Now we can simplify above two equations by applying Snell's two laws as follows

$$1+\Gamma_{TE} = \tau_{TE}$$

$$-\frac{\cos\theta_i}{\eta_1} + \Gamma_{TE}\frac{\cos\theta_r}{\eta_1} = -\frac{\tau_{TE}}{\eta_2}\cos\theta_t$$

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- The above two equations has two unknowns τ_{TE} and Γ_{TE} and it can be solved easily as follows

$$\tau_{TE} = \left(\frac{\cos\theta_i}{\eta_1} - \Gamma_{TE} \frac{\cos\theta_r}{\eta_1} \right) \frac{\eta_2}{\cos\theta_t} \qquad 1 + \Gamma_{TE} = \tau_{TE}$$

- Therefore,

$$1 + \Gamma_{TE} = \left(\frac{\eta_2}{\eta_1} \frac{\cos\theta_i}{\cos\theta_t} - \Gamma_{TE} \frac{\eta_2}{\eta_1} \frac{\cos\theta_r}{\cos\theta_t} \right) \Rightarrow \Gamma_{TE} \left(1 + \frac{\eta_2 \cos\theta_r}{\eta_1 \cos\theta_t} \right) = \left(\frac{\eta_2 \cos\theta_i}{\eta_1 \cos\theta_t} - 1 \right)$$

$$\Rightarrow \Gamma_{TE} \left(\frac{\eta_1 \cos\theta_t + \eta_2 \cos\theta_r}{\eta_1 \cos\theta_t} \right) = \left(\frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_1 \cos\theta_t} \right) \Rightarrow \Gamma_{TE} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_1 \cos\theta_t + \eta_2 \cos\theta_r}$$

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$$\therefore \tau_{TE} = 1 + \Gamma_{TE} = 1 + \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_r} = \frac{\eta_1 \cos\theta_i + \eta_2 \cos\theta_r + \eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_r} = \frac{2\eta_2 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_r}$$

- Hence, the reflection and transmission coefficients for oblique incidence (*Fresnel coefficients)* for perpendicular polarization are given as follows:

$$\Gamma_{TE} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \qquad \tau_{TE} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

- For normal incidence, it is a particular case and put

$$\theta_i = \theta_r = \theta_t = 0$$