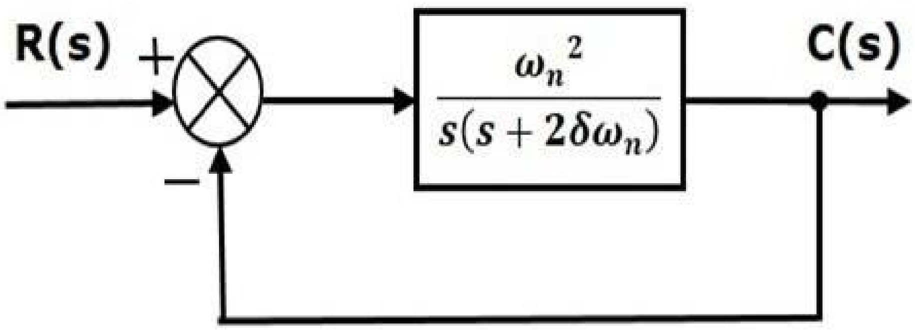
TIME RESPONSE OF SECOND ORDER SYSTEM

Consider the following block diagram of closed loop control system. Here, an open loop transfer function, ω^2 / s(s+2δωn) is connected with a unity negative feedback.



We know that the transfer function of the closed loop control system having unity negative feedback as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Substitute, $G(s) = \frac{\omega_n^2}{s(s+2\delta\omega_n)}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)}{1+\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)} = \frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2}$$

The power of ‘s’ is two in the denominator term. Hence, the above transfer function is of the second order and the system is said to be the **second order system**.

The characteristic equation is -

$$s^2 + 2\delta\omega_n s + \omega_n^2 = 0$$

The roots of characteristic equation are -

$$s = \frac{-2\omega\delta_n \pm \sqrt{(2\delta\omega_n)^2 - 4\omega_n^2}}{2} = \frac{-2(\delta\omega_n \pm \omega_n\sqrt{\delta^2-1})}{2}$$

$$\Rightarrow s = -\delta\omega_n \pm \omega_n\sqrt{\delta^2-1}$$

- The two roots are imaginary when $\delta = 0$.
- The two roots are real and equal when $\delta = 1$.
- The two roots are real but not equal when $\delta > 1$.
- The two roots are complex conjugate when $0 <$

$\delta < 1$. We can write C(s) equation as,

$$C(s) = \left(\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2} \right) R(s)$$

Where,

- **C(s)** is the Laplace transform of the output signal, c(t)
- **R(s)** is the Laplace transform of the input signal, r(t)
- **ω_n** is the natural frequency
- **δ** is the damping ratio.

Follow these steps to get the response (output) of the second order system in the time domain.

Take Laplace transform of the input signal, $r(t)$.

Consider the equation, $C(s) = \left(\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2} \right) R(s)$

Substitute $R(s)$ value in the above equation.

Do partial fractions of $C(s)$ if required.

Apply inverse Laplace transform to $C(s)$.

Step Response of Second Order System

Consider the unit step signal as an input to the second order system. Laplace transform of the unit step signal is,

$$R(s) = \frac{1}{s}$$

We know the transfer function of the second order closed loop control system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Case 1: δ = 0

Substitute, $\delta = 0$ in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - \cos(\omega_n t))\, u(t)$$

So, the unit step response of the second order system when $/delta = 0$ will be a continuous time signal with constant amplitude and frequency.

Case 2: δ = 1

Substitute, $/delta = 1$ in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s + \omega_n)^2}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s + \omega_n)^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s + \omega_n)^2}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

After simplifying, you will get the values of A, B and C as $1, -1 and -\omega_n$ respectively. Substitute these values in the above partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t})u(t)$$

So, the unit step response of the second order system will try to reach the step input in steady state.

Case 3: 0 < δ < 1

We can modify the denominator term of the transfer function as follows −

$$s^2 + 2\delta\omega_n s + \omega_n^2 = \left\{s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2\right\} + \omega_n^2 - (\delta\omega_n)^2$$

$$= (s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)$$

The transfer function becomes,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s\left((s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)\right)}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s\left((s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)\right)} = \frac{A}{s} + \frac{Bs + C}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

After simplifying, you will get the values of A, B and C as $1,\ -1\, and\ -2\delta\omega_n$ respectively. Substitute these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s} - \frac{s + 2\delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

$$C(s) = \frac{1}{s} - \frac{s + \delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)} - \frac{\delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

$$C(s) = \frac{1}{s} - \frac{(s+\delta\omega_n)}{(s+\delta\omega_n)^2 + (\omega_n\sqrt{1-\delta^2})^2} - \frac{\delta}{\sqrt{1-\delta^2}}\left(\frac{\omega_n\sqrt{1-\delta^2}}{(s+\delta\omega_n)^2 + (\omega_n\sqrt{1-\delta^2})^2}\right)$$

Substitute, $\omega_n\sqrt{1-\delta^2}$ as ω_d in the above equation.

$$C(s) = \frac{1}{s} - \frac{(s+\delta\omega_n)}{(s+\delta\omega_n)^2 + \omega_d^2} - \frac{\delta}{\sqrt{1-\delta^2}}\left(\frac{\omega_d}{(s+\delta\omega_n)^2 + \omega_d^2}\right)$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 - e^{-\delta\omega_n t}\cos(\omega_d t) - \frac{\delta}{\sqrt{1-\delta^2}}e^{-\delta\omega_n t}\sin(\omega_d t)\right)u(t)$$

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\left((\sqrt{1-\delta^2})\cos(\omega_d t) + \delta\sin(\omega_d t)\right)\right)u(t)$$

If $\sqrt{1-\delta^2} = \sin(\theta)$, then 'δ' will be cos(θ). Substitute these values in the above equation.

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}(\sin(\theta)\cos(\omega_d t) + \cos(\theta)\sin(\omega_d t))\right)u(t)$$

$$\Rightarrow c(t) = \left(1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)\right)u(t)$$

So, the unit step response of the second order system is having damped oscillations (decreasing amplitude) when ‘δ’ lies between zero and one.

Case 4: δ > 1

We can modify the denominator term of the transfer function as follows −

$$s^2 + 2\delta\omega_n s + \omega_n^2 = \left\{s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2\right\} + \omega_n^2 - (\delta\omega_n)^2$$

$$= (s + \delta\omega_n)^2 - \omega_n^2\left(\delta^2 - 1\right)$$

The transfer function becomes,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\delta\omega_n)^2 - \omega_n^2(\delta^2-1)}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 - \omega_n^2(\delta^2-1)}\right)R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 - (\omega_n\sqrt{\delta^2-1})^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s+\delta\omega_n+\omega_n\sqrt{\delta^2-1})(s+\delta\omega_n-\omega_n\sqrt{\delta^2-1})}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s(s+\delta\omega_n+\omega_n\sqrt{\delta^2-1})(s+\delta\omega_n-\omega_n\sqrt{\delta^2-1})}$$

$$= \frac{A}{s} + \frac{B}{s+\delta\omega_n+\omega_n\sqrt{\delta^2-1}} + \frac{C}{s+\delta\omega_n-\omega_n\sqrt{\delta^2-1}}$$

After simplifying, you will get the values of A, B and C as 1, $\frac{1}{2(\delta+\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}$

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and $\frac{-1}{2(\delta-\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}$ respectively. Substitute these values in above partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s} + \frac{1}{2(\delta+\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\left(\frac{1}{s+\delta\omega_n+\omega_n\sqrt{\delta^2-1}}\right) - \left(\frac{1}{2(\delta-\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\right)\left(\frac{1}{s+\delta\omega_n-\omega_n\sqrt{\delta^2-1}}\right)$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1+\left(\frac{1}{2(\delta+\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\right)e^{-(\delta\omega_n+\omega_n\sqrt{\delta^2-1})t} - \left(\frac{1}{2(\delta-\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\right)e^{-(\delta\omega_n-\omega_n\sqrt{\delta^2-1})t}\right)u(t)$$

Since it is over damped, the unit step response of the second order system when $\delta > 1$ will never reach step input in the steady state.

Impulse Response of Second Order System

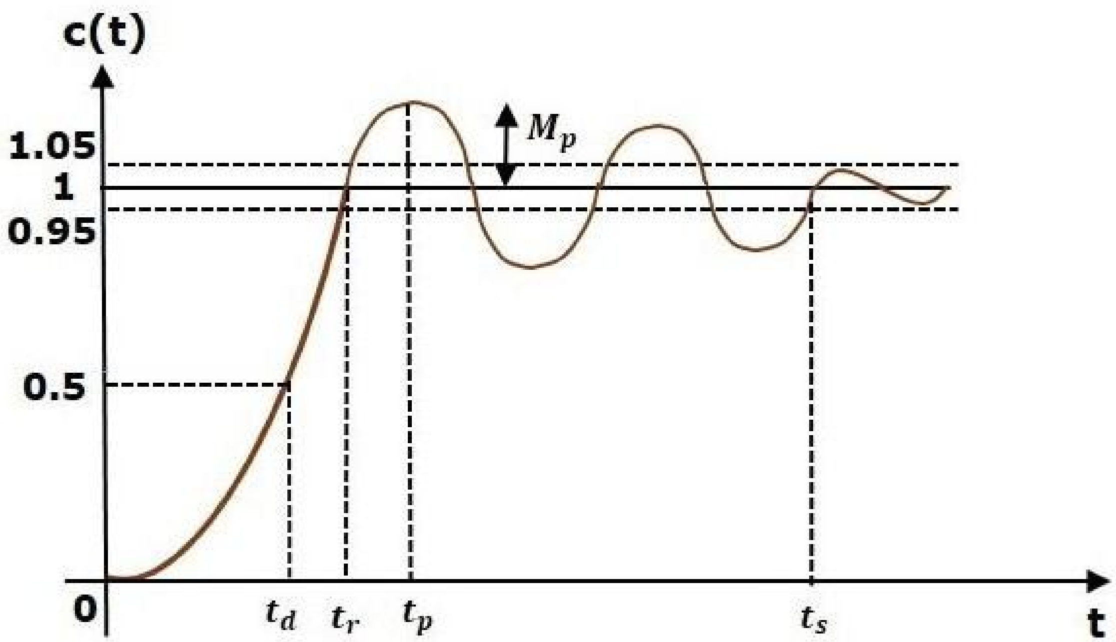
The **impulse response** of the second order system can be obtained by using any one of these two methods.

- Follow the procedure involved while deriving step response by considering the value of R(s) as 1 instead of 1/s.
- Do the differentiation of the step response.

The following table shows the impulse response of the second order system for 4 cases of the damping ratio.

Condition of Damping ratio	Impulse response for t ≥ 0
$\delta = 0$	$\omega_n \sin(\omega_n t)$
$\delta = 1$	$\omega_n^2 t e^{-\omega_n t}$
$0 < \delta < 1$	$\left(\frac{\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t)$
$\delta > 1$	$\left(\frac{\omega_n}{2\sqrt{\delta^2-1}}\right)\left(e^{-(\delta\omega_n-\omega_n\sqrt{\delta^2-1})t} - e^{-(\delta\omega_n+\omega_n\sqrt{\delta^2-1})t}\right)$

In this chapter, let us discuss the time domain specifications of the second order system. The step response of the second order system for the underdamped case is shown in the following figure.



All the time domain specifications are represented in this figure. The response up to the settling time is known as transient response and the response after the settling time is known as steady state response.

Delay Time

It is the time required for the response to reach **half of its final value** from the zero instant. It is denoted by tdtd.

Consider the step response of the second order system for t ≥ 0, when 'δ' lies between zero and one.

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

The final value of the step response is one.

Therefore, at $t = t_d$, the value of the step response will be 0.5. Substitute, these values in the above equation.

$$c(t_d) = 0.5 = 1 - \left(\frac{e^{-\delta\omega_n t_d}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_d + \theta)$$

$$\Rightarrow \left(\frac{e^{-\delta\omega_n t_d}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_d + \theta) = 0.5$$

By using linear approximation, you will get the **delay time t_d** as

$$t_d = \frac{1 + 0.7\delta}{\omega_n}$$

Rise Time

It is the time required for the response to rise from **0% to 100% of its final value**. This is applicable for the **under-damped systems**. For the over-damped systems, consider the duration from 10% to 90% of the final value. Rise time is denoted by $\mathbf{t_r}$.

At t = t_1 = 0, c(t) = 0.

We know that the final value of the step response is one.Therefore, at t=t2, the value of step response is one. Substitute, these values in the following equation.

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

$$c(t_2) = 1 = 1 - \left(\frac{e^{-\delta\omega_n t_2}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_2 + \theta)$$

$$\Rightarrow \left(\frac{e^{-\delta\omega_n t_2}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_2 + \theta) = 0$$

$$\Rightarrow \sin(\omega_d t_2 + \theta) = 0$$

$$\Rightarrow \omega_d t_2 + \theta = \pi$$

$$\Rightarrow t_2 = \frac{\pi - \theta}{\omega_d}$$

Substitute t_1 and t_2 values in the following equation of **rise time**,

$$t_r = t_2 - t_1$$

$$\therefore\ t_r = \frac{\pi - \theta}{\omega_d}$$

From above equation, we can conclude that the rise time t_r and the damped frequency ω_d are inversely proportional to each other.

Peak Time

It is the time required for the response to reach the **peak value** for the first time. It is denoted by t_p. At $t=t_p$ the first derivate of the response is zero.

We know the step response of second order system for under-damped case is

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

Differentiate $c(t)$ with respect to 't'.

$$\frac{\mathrm{d}c(t)}{\mathrm{d}t} = -\left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\omega_d\cos(\omega_d t + \theta) - \left(\frac{-\delta\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

Differentiate $c(t)$ with respect to 't'.

$$\frac{\mathrm{d}c(t)}{\mathrm{d}t} = -\left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\omega_d\cos(\omega_d t + \theta) - \left(\frac{-\delta\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

Substitute, $t = t_p$ and $\frac{\mathrm{d}c(t)}{\mathrm{d}t} = 0$ in the above equation.

$$0 = -\left(\frac{e^{-\delta\omega_n t_p}}{\sqrt{1-\delta^2}}\right)\left[\omega_d\cos(\omega_d t_p + \theta) - \delta\omega_n\sin(\omega_d t_p + \theta)\right]$$

$$\Rightarrow \omega_n\sqrt{1-\delta^2}\cos(\omega_d t_p + \theta) - \delta\omega_n\sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sqrt{1-\delta^2}\cos(\omega_d t_p + \theta) - \delta\sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sin(\theta)\cos(\omega_d t_p + \theta) - \cos(\theta)\sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sin(\theta - \omega_d t_p - \theta) = 0$$

$$\Rightarrow sin(-\omega_d t_p) = 0 \Rightarrow -\sin(\omega_d t_p) = 0 \Rightarrow sin(\omega_d t_p) = 0$$

$$\Rightarrow \omega_d t_p = \pi$$
$$\Rightarrow t_p = \frac{\pi}{\omega_d}$$

From the above equation, we can conclude that the peak time t_p and the damped frequency ω_d are inversely proportional to each other.

Peak Overshoot

Peak overshoot $\mathbf{M_p}$ is defined as the deviation of the response at peak time from the final value of response. It is also called the **maximum overshoot**.

Mathematically, we can write it as

$$Mp = c(t_p) - c(\infty)$$

Where, $c(t_p)$ is the peak value of the response, $c(\infty)$ is the final (steady state) value of the response

At t=tp, the response c(t) is -

$$c(t_p) = 1 - \left(\frac{e^{-\delta\omega_n t_p}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_p + \theta)$$

Substitute, $t_p = \frac{\pi}{\omega_d}$ in the right hand side of the above equation.

$$c(t_P) = 1 - \left(\frac{e^{-\delta\omega_n\left(\frac{\pi}{\omega_d}\right)}}{\sqrt{1-\delta^2}}\right)\sin\left(\omega_d\left(\frac{\pi}{\omega_d}\right)+\theta\right)$$

$$\Rightarrow c(t_p) = 1 - \left(\frac{e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}}{\sqrt{1-\delta^2}}\right)(-\sin(\theta))$$

We know that

$$\sin(\theta) = \sqrt{1-\delta^2}$$

So, we will get $c(t_p)$ as

$$c(t_p) = 1 + e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}$$

Substitute the values of $c(t_p)$ and $c(\infty)$ in the peak overshoot equation.

$$M_p = 1 + e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)} - 1$$

$$\Rightarrow M_p = e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}$$

Percentage of peak overshoot % M_p can be calculated by using this formula.

$$\%M_p = \frac{M_p}{c(\infty)} \times 100\%$$

From the above equation, we can conclude that the percentage of peak overshoot %Mp will decrease if the damping ratio δ increases.

Settling time

It is the time required for the response to reach the steady state and stay within the specified tolerance bands around the final value. In general, the tolerance bands are 2% and 5%. The settling time is denoted by ts.

The settling time for 5% tolerance band is –

$$t_s = \frac{3}{\delta\omega_n} = 3\tau$$

The settling time for 2% tolerance band is –

$$t_s = \frac{4}{\delta\omega_n} = 4\tau$$

Where, τ is the time constant and is equal to $1/\delta\omega_n$.

- Both the settling time ts and the time constant τ are inversely proportional to the damping ratio δ.

Both the settling time ts and the time constant τ are independent of the system gain. That means even the system gain changes, the settling time ts and time constant τ will never change.