TIME RESPONSE ANALYSIS

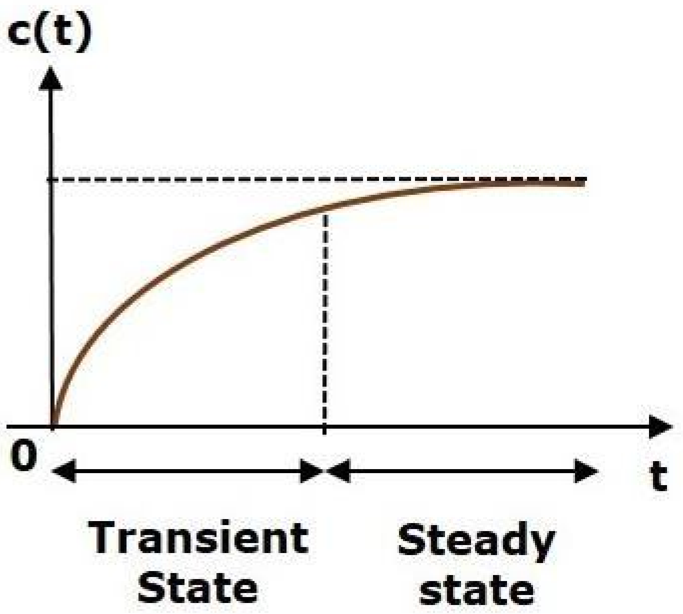
We can analyze the response of the control systems in both the time domain and the frequency domain. We will discuss frequency response analysis of control systems in later chapters. Let us now discuss about the time response analysis of control systems.

What is Time Response?

If the output of control system for an input varies with respect to time, then it is called the **time response** of the control system. The time response consists of two parts.

- Transient response
- Steady state response

The response of control system in time domain is shown in the following figure



Here, both the transient and the steady states are indicated in the figure. The responses corresponding to these states are known as transient and steady state responses.

Mathematically, we can write the time response c(t) as

$$c(t) = c_{tr}(t) + c_{ss}(t)$$

Where,

- $c_{tr}(t)$ is the transient response
- $c_{ss}(t)$ is the steady state response

Transient Response

After applying input to the control system, output takes certain time to reach steady state. So, the output will be in transient state till it goes to a steady state. Therefore, the response of the control system during the transient state is known as **transient response**.

The transient response will be zero for large values of ‘t’. Ideally, this value of ‘t’ is infinity and practically, it is five times constant.

Mathematically, we can write it as

$$\lim_{t\to\infty} c_{tr}(t) = 0$$

Steady state Response

The part of the time response that remains even after the transient response has zero value for large values of ‘t’ is known as **steady state response**. This means, the transient response will be zero even during the steady state.

Example

Let us find the transient and steady state terms of the time response of the control $c(t) = 10 + 5e^{-t}$ system

Here, the second term $5e^{-t}$ will be zero as **t** denotes infinity. So, this is the **transient term**. And the first term 10 remains even as **t** approaches infinity. So, this is the **steady state term**.

Standard Test Signals

The standard test signals are impulse, step, ramp and parabolic. These signals are used to know the performance of the control systems using time response of the output.

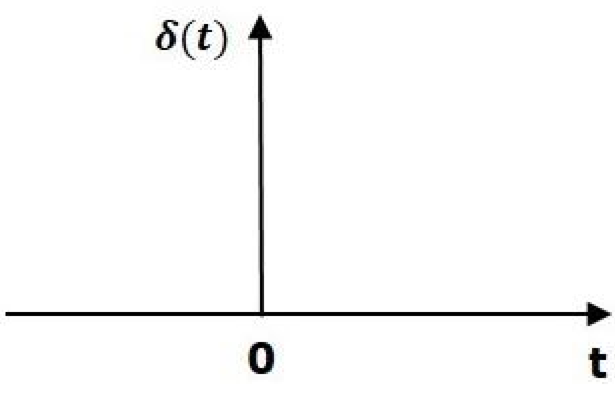
Unit Impulse Signal

A unit impulse signal, δ(t) is defined as

$$\delta(t) = 0 \text{ for } t \neq 0$$

$$\text{and } \int_{0^-}^{0^+} \delta(t)dt = 1$$

The following figure shows unit impulse signal.



So, the unit impulse signal exists only at‘t’ is equal to zero. The area of this signal under small interval of time around‘t’ is equal to zero is one. The value of unit impulse signal is zero for all other values of‘t’.

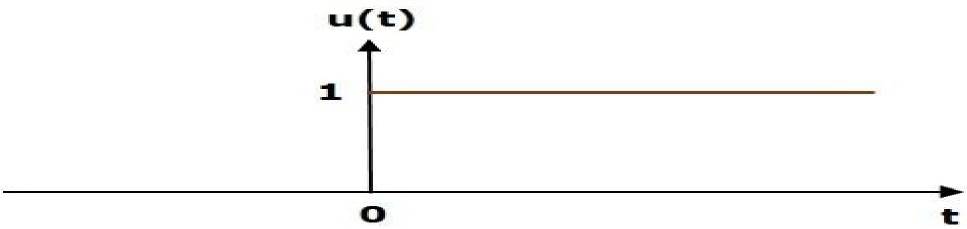
Unit Step Signal

A unit step signal, u(t) is defined as

$$u(t) = 1; t \geq 0$$

$$= 0; t < 0$$

Following figure shows unit step signal.



So, the unit step signal exists for all positive values of‘t’ including zero. And its value is one during this interval. The value of the unit step signal is zero for all negative values of‘t’.

Unit Ramp Signal

A unit ramp signal, r (t) is defined as

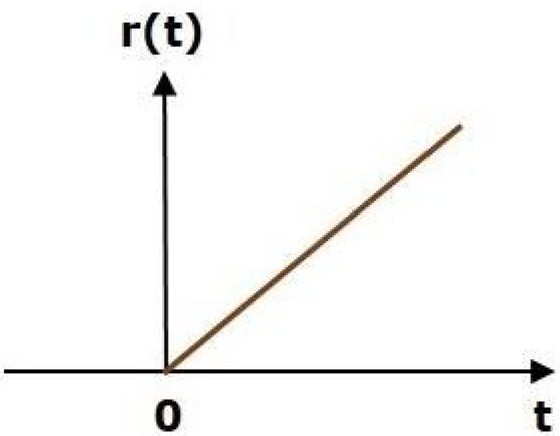
$$r(t) = t; t \geq 0$$

$$= 0; t < 0$$

We can write unit ramp signal, $r(t)$ in terms of unit step signal, $u(t)$ as

$$r(t) = tu(t)$$

Following figure shows unit ramp signal.



So, the unit ramp signal exists for all positive values of‘t’ including zero. And its value increases linearly with respect to‘t’ during this interval. The value of unit ramp signal is zero for all negative values of‘t’.

Unit Parabolic Signal

A unit parabolic signal, p(t) is defined as,

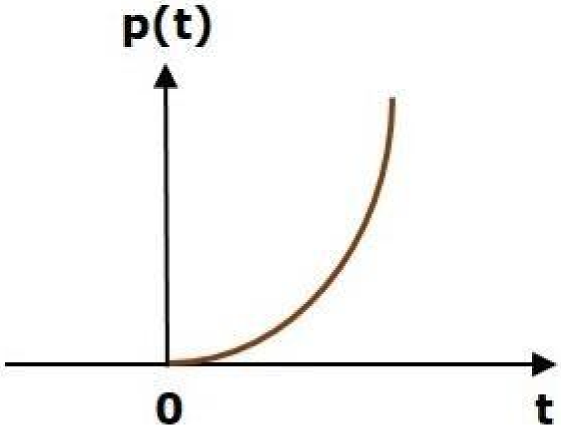
$$p(t) = \frac{t^2}{2}; t \geq 0$$

$$= 0; t < 0$$

We can write unit parabolic signal, $p(t)$ in terms of the unit step signal, $u(t)$ as,

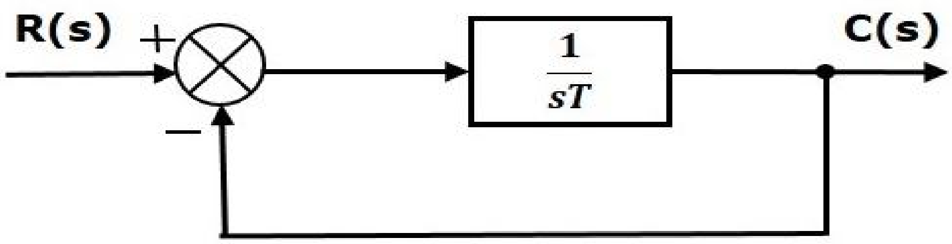
$$p(t) = \frac{t^2}{2} u(t)$$

The following figure shows the unit parabolic signal.



So, the unit parabolic signal exists for all the positive values of‘t’ including zero. And its value increases non-linearly with respect to‘t’ during this interval. The value of the unit parabolic signal is zero for all the negative values of‘t’.

let us discuss the time response of the first order system. Consider the following block diagram of the closed loop control system. Here, an open loop transfer function, 1/sT is connected with a unity negative feedback.



We know that the transfer function of the closed loop control system has unity negative feedback as,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

Substitute, $G(s) = \frac{1}{sT}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{sT}}{1 + \frac{1}{sT}} = \frac{1}{sT + 1}$$

The power of s is one in the denominator term. Hence, the above transfer function is of the first order and the system is said to be the **first order system**.

We can re-write the above equation as

$$C(s) = \left(\frac{1}{sT+1}\right) R(s)$$

Where,

- $\mathbf{C(s)}$ is the Laplace transform of the output signal c(t),
- $\mathbf{R(s)}$ is the Laplace transform of the input signal r(t), and
- **T** is the time constant.

Follow these steps to get the response (output) of the first order system in the time domain.

- Take the Laplace transform of the input signal $r(t)$.
- Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$
- Substitute $R(s)$ value in the above equation.
- Do partial fractions of $C(s)$ if required.
- Apply inverse Laplace transform to $C(s)$.

Impulse Response of First Order System

Consider the **unit impulse signal** as an input to the first order system.

So, r(t)=δ(t)

Apply Laplace transform on

both the sides. R(s) =1

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = 1$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)(1) = \frac{1}{sT+1}$$

Rearrange the above equation in one of the standard forms of Laplace transforms.

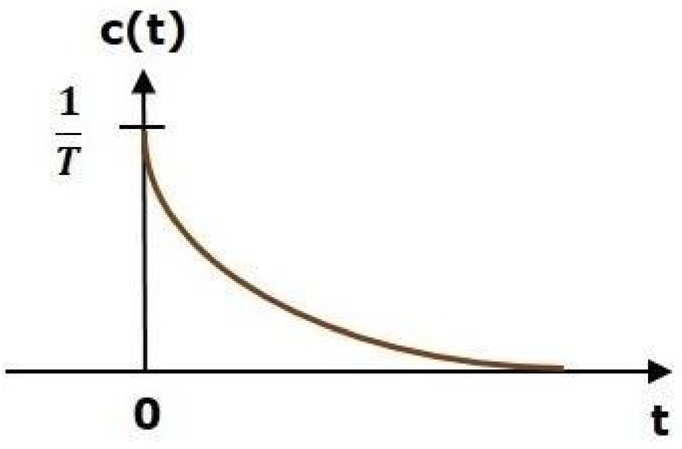
$$C(s) = \frac{1}{T\left(s+\frac{1}{T}\right)} \Rightarrow C(s) = \frac{1}{T}\left(\frac{1}{s+\frac{1}{T}}\right)$$

Applying Inverse Laplace Transform on both the sides,

$$c(t) = \frac{1}{T}e^{\left(-\frac{t}{T}\right)}u(t)$$

$$c(t) = \frac{1}{T}e^{\left(-\frac{t}{T}\right)}u(t)$$

The unit impulse response is shown in the following figure.



The **unit impulse response**, c(t) is an exponential decaying signal for positive values of 't' and it is zero for negative values of 't'.

Step Response of First Order System

Consider the **unit step signal** as an input to

first order system. So, r(t)=u(t)

$$R(s) = \frac{1}{s}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right)R(s)$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s}\right) = \frac{1}{s\left(sT+1\right)}$$

Do partial fractions of C(s).

$$C(s) = \frac{1}{s\left(sT+1\right)} = \frac{A}{s} + \frac{B}{sT+1}$$

$$\Rightarrow \frac{1}{s\left(sT+1\right)} = \frac{A\left(sT+1\right)+Bs}{s\left(sT+1\right)}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$1=A(sT+1)+Bs$$

By equating the constant terms on both the sides, you will get A = 1. Substitute, A = 1 and equate the coefficient of the **s** terms on both the sides.

$$0=T+B$$

$$\Rightarrow B=-T$$

Substitute, A = 1 and B = −T in partial fraction expansion of C(s)

$$C(s)=\frac{1}{s}-\frac{T}{sT+1}=\frac{1}{s}-\frac{T}{T\left(s+\frac{1}{T}\right)}$$

$$\Rightarrow C(s)=\frac{1}{s}-\frac{1}{s+\frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t)=\left(1-e^{-\left(\frac{t}{T}\right)}\right)u(t)$$

The **unit step response**, c(t) has both the transient and the steady state terms.

The transient term in the unit step response is -

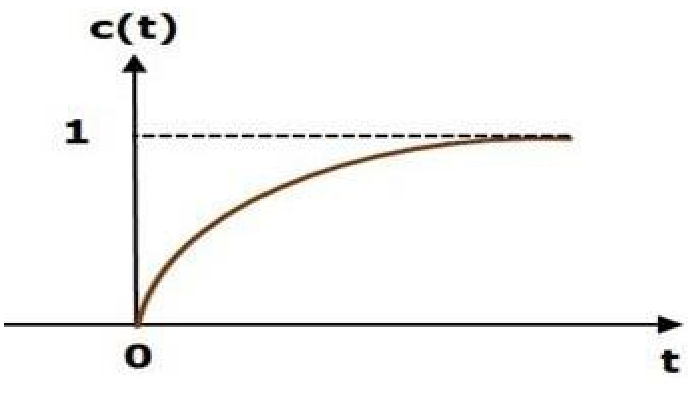
$$c_{tr}(t)=-e^{-\left(\frac{t}{T}\right)}u(t)$$

The steady state term in the unit step

$$c_{ss}(t)=u(t)$$

response is – The following figure

shows the unit step response



The value of the **unit step response, c(t)** is zero at t = 0 and for all negative values of t. It is gradually increasing from zero value and finally reaches to one in steady state. So, the steady state value depends on the magnitude of the input.

Ramp Response of First Order System

Consider the **unit ramp signal** as an input to the first order system.

So,r(t)=t u(t)

Apply Laplace transform on both the sides.

$$R(s) = \frac{1}{s^2}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = \frac{1}{s^2}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s^2}\right) = \frac{1}{s^2(sT+1)}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{1}{s^2(sT+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{sT+1}$$

$$\Rightarrow \frac{1}{s^2(sT+1)} = \frac{A(sT+1) + Bs(sT+1) + Cs^2}{s^2(sT+1)}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$1 = A(sT+1) + Bs(sT+1) + Cs^2$$

By equating the constant terms on both the sides, you will

get A = 1. Substitute, A = 1 and equate the coefficient of the

s terms on both the sides.

$$0=T+B \Rightarrow B=-T$$

Similarly, substitute B = −T and equate the coefficient of s^2 terms on both the sides. You will get $C=T^2$

Substitute A = 1, B = −T and $C=T^2$ in the partial fraction expansion of C(s).

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{sT+1} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{T\left(s+\frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s+\frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(t - T + Te^{-\left(\frac{t}{T}\right)}\right) u(t)$$

The **unit ramp response**, c(t) has both the transient and the steady state terms.

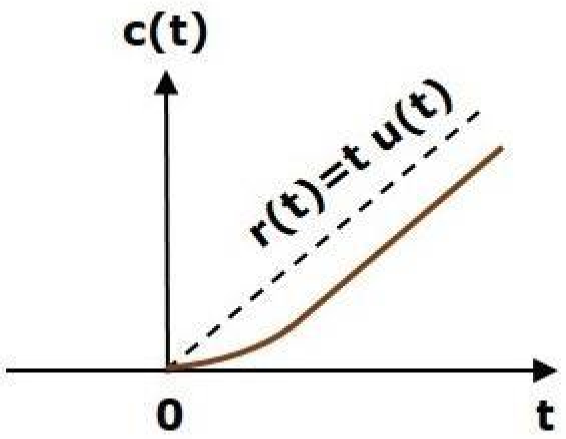
The transient term in the unit ramp response is

$$c_{tr}(t) = Te^{-\left(\frac{t}{T}\right)}u(t)$$

The steady state term in the unit ramp response is –

$$c_{ss}(t) = (t - T)u(t)$$

The figure below is the unit ramp response:



The **unit ramp response**, c(t) follows the unit ramp input signal for all positive values of t. But, there is a deviation of T units from the input signal.

Parabolic Response of First Order System

Consider the **unit parabolic signal** as an input to the first order system.

$$\text{So, } r(t) = \frac{t^2}{2}u(t)$$

Apply Laplace transform on both the sides.

$$R(s) = \frac{1}{s^3}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right)R(s)$

Substitute $R(s) = \frac{1}{s^3}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s^3}\right) = \frac{1}{s^3(sT+1)}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{1}{s^3(sT+1)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{sT+1}$$

After simplifying, you will get the values of A, B, C and D as 1, $-T, T^2 \, and \, -T^3$ respectively. Substitute these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s^3} - \frac{T}{s^2} + \frac{T^2}{s} - \frac{T^3}{sT+1} \Rightarrow C(s) = \frac{1}{s^3} - \frac{T}{s^2} + \frac{T^2}{s} - \frac{T^2}{s+\frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(\frac{t^2}{2} - Tt + T^2 - T^2 e^{-\left(\frac{t}{T}\right)}\right) u(t)$$

The **unit parabolic response**, c(t) has both the transient and the steady state terms.

The transient term in the unit parabolic response is

$$C_{tr}(t) = -T^2 e^{-\left(\frac{t}{T}\right)} u(t)$$

The steady state term in the unit parabolic response is

$$C_{ss}(t) = \left(\frac{t^2}{2} - Tt + T^2\right) u(t)$$

From these responses, we can conclude that the first order control systems are not stable with the ramp and parabolic inputs because these responses go on increasing even at infinite amount of time. The first order control systems are stable with impulse and step inputs because these responses have bounded output. But, the impulse response doesn’t have steady state term. So, the step signal is widely used in the time domain for analyzing the control systems from their responses.