

Module 3 : Sampling & Reconstruction

Lecture 23 : Low pass filter

Objectives:

Scope of lecture:

In the previous lecture we mentioned that the ideal low pass filter can be used to recover the original continuous time signal from its samples. In this lecture, we will derive its impulse response, and see how exactly the original signal can be recovered. We'll be covering:

- The impulse response of an ideal low pass filter.
- Reconstruction of a band-limited signal by a low pass filter.
- Problems with the ideal low pass filter.

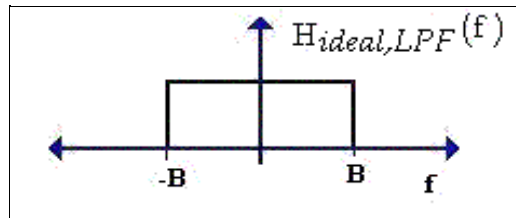
The Ideal Low-Pass Filter

In this lecture, we examine the Ideal low pass filter and the process of reconstruction of a Band-limited signal.

Let us first see another way of interpreting the action of the Ideal low-pass filter.

IMPULSE RESPONSE OF IDEAL LOW PASS FILTER:

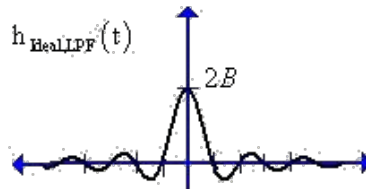
The Frequency response (the Fourier transform of the impulse response of an LSI system is also called its frequency response) of an ideal low pass filter which allows a bandwidth B , is a rectangle extending from $-B$ to $+B$, having a constant height as shown in the figure.



Lets look at the Impulse Response of this Ideal low pass filter, taking its height in $[-B, B]$ to be 1. Using the formula for inverse Fourier Transform we have :

$$\begin{aligned} \text{Impulse Response} &= \int_{-B}^B e^{j\omega t} df \\ &= \frac{2B \sin \pi(2Bt)}{\pi(2Bt)} \\ &= 2B \text{Sinc}(2Bt) \quad (\text{note that } \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}) \end{aligned}$$

Thus the impulse response of an ideal low pass filter turns out to be a **Sinc function**, which looks like:



Reconstruction of a signal by low pass filter :

Consider a signal $x(t)$ having bandwidth less than B .

We sample $x(t)$ at a rate $2B$ and pass $\tilde{x}(t) = \sum_{-\infty}^{\infty} \frac{1}{2B} x(\frac{n}{2B}) \delta(t - \frac{n}{2B})$ into an Ideal low-pass filter of bandwidth B . Assume $T_0 = \frac{1}{2B}$.

The signal $\mathbf{x(t)}$ and the signal ($\tilde{\mathbf{x}}(t)$) obtained by multiplying the signal by a periodic train of impulses, separated by T_0 , having strength 1 are shown below.



What happens when $\tilde{\mathbf{x}}(t)$ is fed into the LSI system?

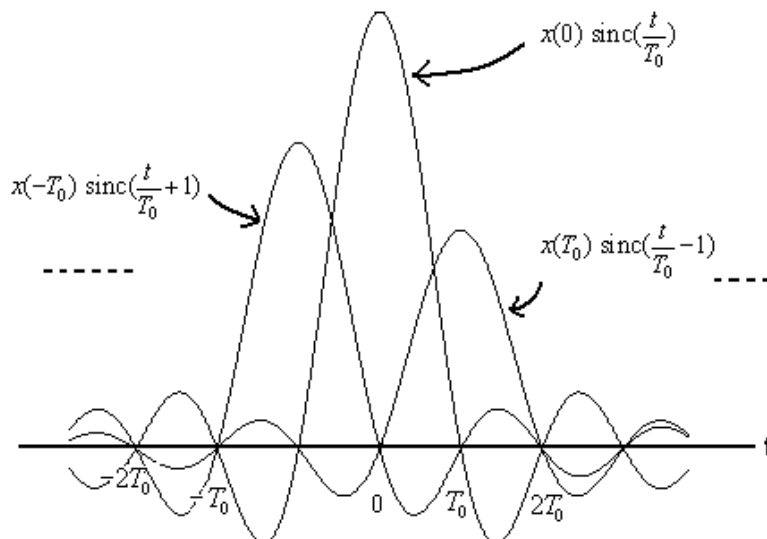
Lets look at the convolution of the impulse response $\mathbf{h(t)}$ of the Ideal low-pass filter with $\tilde{\mathbf{x}}(t)$

$$\begin{aligned} \tilde{\mathbf{x}}(t) * \mathbf{h}(t) &= \left\{ \sum_{n=-\infty}^{\infty} T_0 x(nT_0) \delta(t - nT_0) \right\} * \mathbf{h}(t) \\ &= \sum_{n=-\infty}^{\infty} T_0 x(nT_0) \mathbf{h}(t - nT_0) \end{aligned}$$

where we have seen $\mathbf{h}(t) = 2B \operatorname{sinc}(2Bt) = \frac{1}{T_0} \operatorname{sinc}\left(\frac{t}{T_0}\right)$.

$$\therefore \tilde{\mathbf{x}}(t) * \mathbf{h}(t) = \sum_{n=-\infty}^{\infty} x(nT_0) \operatorname{sinc}\left(\frac{t}{T_0} - n\right)$$

When $\tilde{\mathbf{x}}(t)$ is passed through a Low pass filter, the output which is the reconstructed signal is nothing but the sum of copies of the impulse response $\mathbf{h(t)}$ shifted by integral multiples of T_0 and multiplied by the value of $\mathbf{x(t)}$ at the corresponding integral multiple of T_0 . Also observe that the $\mathbf{h(t)}$ is zero at all sample points (which are integral multiples of T_0) except at zero. Thus, the reconstruction of $\mathbf{x(t)}$ can be visualized as a sum of the following signals :



Problems with the IDEAL LOW PASS FILTER

- **It is infinitely Non-Causal:**

The impulse response of the ideal low pass filter extends to $-\infty$. If the impulse response is denoted by $\mathbf{h(t)}$, the output signal $\mathbf{y(t)}$ corresponding to input signal $\mathbf{x(t)}$ is given by :

$$y(t) = \int_{-\infty}^{\infty} x(\lambda) h(t - \lambda) d\lambda$$

The value of \mathbf{y} at any \mathbf{t} depends on values of \mathbf{x} all the way to $+\infty$ if $\mathbf{h(t)}$ extends to $-\infty$. Thus realization in real time is not possible for an Ideal low-pass filter. In other words, unless one knows the entire $\tilde{x}(t)$, reconstruction cannot be done.

Note if $\mathbf{h(t)}$ had been finitely non-causal (say zero for all t less than some $-t_0$), then real time realization would have been possible subject to a time-delay (of t_0).

- **It is unstable:**

It can be shown that $\int_{-\infty}^{\infty} |h(t)| dt$ diverges.

Challenging Problem: Prove that Ideal filter is unstable.

Proof:

$$\begin{aligned} & \int_{-\infty}^{\infty} 2B |\text{Sinc}(2Bt)| dt, \quad (\text{put } 2Bt = y) \\ &= 2 \int_0^{\infty} \frac{|\sin y|}{|y|} dy = 2 \sum_{n=1}^{\infty} \int_{(n-1)\pi}^{n\pi} \frac{|\sin y|}{|y|} dy \end{aligned}$$

which is greater than

$$\sum_{n=1}^{\infty} \frac{4}{n\pi}$$

and we know that this series diverges. Hence it is established that the Ideal Low Pass Filter is unstable.

This implies that bounded input does not imply bounded output. Thus if we build an oscillator with Ideal Low pass Filter a bounded input may result in an unstable output.

- **The system is not rational:**

That means, it is not exactly realizable with simple well known elements .

We will get back to how these problems are tackled a little later. In the next lecture, we move on to the problem of impulses not being physically realizable.

Conclusion:

In this lecture you have learnt:

- Impulse response of an ideal low pass filter turns out to be a Sinc function.
- When sampled signal is passed through a low pass filter, reconstructed output signal is nothing but the sum of copies of impulse response $h(t)$ shifted by integral multiples of T_0 and multiplied by the value of $x(t)$ at the corresponding integral multiple of T_0 .
- Problems with the ideal low pass filter :
 1. It is infinitely non-causal.
 2. It is unstable.
 3. It is not a rational system.

Congratulations, you have finished Lecture 23.