UNIT - 1

Discrete Unit Impulse and Step Signals:

The discrete **unit impulse signal** is defined:

$$x[n] = \delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

Useful as a **basis** for analyzing other signals

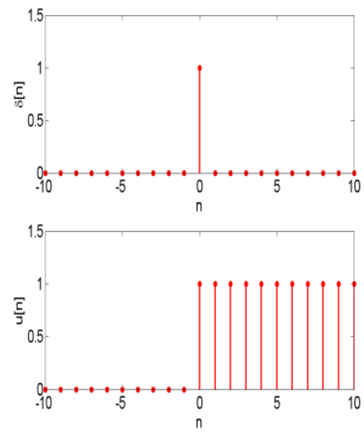
The discrete **unit step signal** is defined:

$$x[n] = u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

Note that the unit impulse is the first difference (derivative) of the step signal

$$\delta[n] = u[n] - u[n-1]$$

Similarly, the unit step is the running sum (integral) of the unit impulse.



Continuous Unit Impulse and Step Signals:

The continuous **unit impulse signal** is defined:

$$x(t) = \delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

Note that it is discontinuous at t=0

The arrow is used to denote area, rather than actual value

Again, useful for an infinite basis

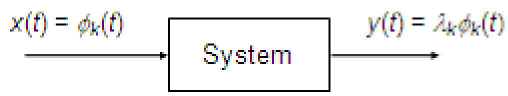
The continuous **unit step signal** is defined:

$$x(t) = u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

$$x(t) = u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

Why is Fourier Theory Important ?

For a particular system, what signals $f_k(t)$ have the property that:



Then $f_k(t)$ is an eigenfunction with eigenvalue l_k

If an input signal can be decomposed as

$$x(t) = \Sigma_k a_k \phi_k(t)$$

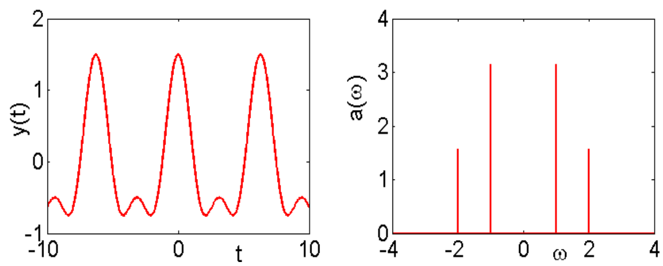
Then the response of an LTI system is

$$y(t) = \Sigma_k a_k \lambda_k \phi_k(t)$$

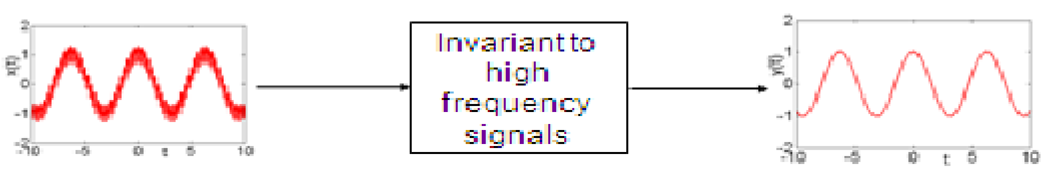
For an LTI system, $f_k(t) = e^{st}$ where $s \in C$, are eigenfunctions.

Fourier transforms map a time-domain signal into a frequency domain signal

Simple interpretation of the frequency content of signals in the frequency domain (as opposed to time).



Design systems to filter out high or low frequency components. Analyse systems in frequency domain.



If F{x(t)} = X(jw) w is the frequency

Then F{x'(t)} = jwX(jw)

So solving a differential equation is transformed from a calculus operation in the time domain into an algebraic operation in the frequency domain (see Laplace transform)

Example $$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = 0$$

becomes $$-\omega^2 Y(j\omega) + j2\omega Y(j\omega) + 3Y(j\omega) = 0$$

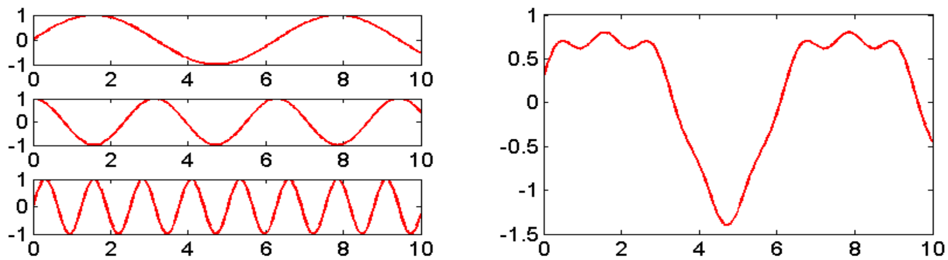
and is solved for the roots w (N.B. complementary equations):

$$-\omega^2 + j2\omega + 3 = 0$$

and we take the inverse Fourier transform for those w.

Fourier Series and Fourier Basis Functions:

The theory derived for LTI convolution, used the concept that any input signal can represented as a linear combination of shifted impulses (for either DT or CT signals) . These are known as continuous-time Fourier series. The bases are scaled and shifted sinusoidal signals, which can be represented as complex exponentials.



Periodic Signals & Fourier Series:

A periodic signal has the property x(t) = x(t+T), T is the fundamental period, w_0 = 2p/T is the fundamental frequency. Two periodic signals include:

$$x(t) = \cos(\omega_0 t)$$
$$x(t) = e^{j\omega_0 t}$$

For each periodic signal, the Fourier basis the set of harmonically related complex exponentials:
$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t} \qquad k = 0, \pm 1, \pm 2, \ldots$$

Thus the Fourier series is of the form: $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

k=0 is a constant

k=+/-1 are the fundamental/first harmonic components

k=+/-N are the N^{th} harmonic components

Fourier Series Representation of a CT Periodic Signal:

Given that a signal has a Fourier series representation, we have to find $\{a_k\}_k$. Multiplying through by $e^{-jn\omega_0 t}$

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}dt$$

Using Euler's formula for the complex exponential integral

$$\int_0^T e^{j(k-n)\omega_0 t}dt = \int_0^T \cos((k-n)\omega_0 t)dt + j\int_0^T \sin((k-n)\omega_0 t)dt$$

It can be shown that

$$\int_0^T e^{j(k-n)\omega_0 t}dt = \begin{cases} T & k = n \\ 0 & k \neq n \end{cases}$$

Therefore
$$a_n = \tfrac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt$$

which allows us to determine the coefficients. Also note that this result is the same if we integrate over any interval of length T (not just [0,T]), denoted by

$$a_n = \tfrac{1}{T}\int_T x(t)e^{-jn\omega_0 t}dt$$

$$\int_T |x(t)|dt < \infty$$

To summarise, if x(t) has a Fourier series representation, then the pair of equations that defines the Fourier series of a periodic, continuous-time signal:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

$$a_k = \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \tfrac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

Example 1: Fourier Series $\sin(w_0 t)$

The fundamental period of $\sin(w_0 t)$ is w_0

By inspection we can write:

$$\sin(\omega_0 t) = \tfrac{1}{2j}e^{j\omega_0 t} - \tfrac{1}{2j}e^{-j\omega_0 t}$$

So $a_1 = 1/2j$, $a_{-1} = -1/2j$ and $a_k = 0$ otherwise

The magnitude and angle of the Fourier coefficients are: