VELOCITY POTENTIAL FUNCTION AND STREAM FUNCTION

Velocity Potential Functions. It is defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction. It is defined by ϕ (Phi). Mathematically, the velocity, potential is defined as $\phi = \int f(s, y, z)$ for steady flow such that

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial x} \\ v &= -\frac{\partial \phi}{\partial y} \\ w &= -\frac{\partial \phi}{\partial z} \end{aligned}\right\} \quad \text{............(1)}$$

where u, v and w are the components of velocity in x,y and z directions respectively.

The velocity components in cylindrical polar co-ordinates in terms of velocity potential functions are given by

$$\left.\begin{aligned} u_r &= \frac{\partial \phi}{\partial r} \\ u_\theta &= \frac{1}{r}\frac{\partial \phi}{\partial \theta} \end{aligned}\right\} \qquad \text{...............(1A)}$$

where u_r = velocity component in radial direction (i.e., in r direction)

and u_θ = velocity component in tangential direction (i.e., in θ direction)

The continuity equation for an incompressible steady flow is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

Substituting the values of u , v and w from equation (5.9) we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right) = 0$$

or

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \text{..........(2)}$$

equation (2) is a Laplace equation.

For two-dimension case, equation (1) reduces to $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ ....(5.11)

If any value of ϕ that satisfies the Laplace equation, will correspond to some case of fluid flow.

Properties of the Potential Function. The rotational components* are given by

$$\omega_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

$$\omega_y = \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right]$$

$$\omega_x = \frac{1}{2}\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right]$$

Substituting the values of u, v and w from equation (1A) in the above rotational components, we get

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial x}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x}\right]$$

$$\omega_y = \frac{1}{2}\left[\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial z}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial z \partial x} + \frac{\partial^2 \phi}{\partial x \partial z}\right]$$

$$\omega_x = \frac{1}{2}\left[\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial y}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial y \partial z} + \frac{\partial^2 \phi}{\partial z \partial y}\right]$$

If ϕ is a continuous function, then $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$; $\frac{\partial^2 \phi}{\partial z \partial x} = \frac{\partial^2 \phi}{\partial x \partial z}$; etc.

$\therefore \qquad \omega_z = \omega_y = \omega_x = 0$

When rotational components are zero, the flow is called ir-rotational. Hence the properties of the potential function are:

1. If velocity potential (ϕ) exists, the flow should be irrotational.
2. If velocity potential (ϕ) satisfies the Laplace equation, it represents the possible steady incompressible irrotational flow.

Stream Function. It is defined as the scalar function of space and time ; such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction. It is denoted by ψ (Psi) and defined only for two-dimensional flow. Mathematically, for steady flow it is defined as $\psi = f(x,y)$ such that

$$\left.\begin{aligned}\frac{\partial\psi}{\partial x} &= v \\ \text{and} \qquad \frac{\partial\psi}{\partial y} &= -u\end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots..(1)$$

The velocity components in cylindrical polar co-ordinates in terms of stream function are given as

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \text{ and } u_\theta = -\frac{\partial\psi}{\partial r} \qquad \ldots\ldots\ldots\ldots\ldots(1A)$$

where u_r = radial velocity and u_θ =tangential velocity

The continuity equation for two-dimensional flow is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

Substituting the values of u and v from equation (5.12), we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x}\right) = \text{or} - \frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial x\partial y} = 0$$

Hence existence of ψ means a possible case of fluid flow. The flow may be rotational or irrotational.

The rotational component ω_z is given by $\omega_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$

Substituting the values of u and v from equation (5.12) in the above rotational component, we get

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial\psi}{\partial y}\right)\right] = \frac{1}{2}\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right]$$

For irrotational flow, ω_z =0. Hence above equation becomes as $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$

Which is Laplace equation for ψ .

The Properties of stream function (ψ) are:

1. If stream function (ψ) exists, it is a possible case of fluid which may be rotational or irrotational.
2. If stream function (ψ) satisfies the Laplace equation, it is possible case of an irrotational flow.

Equipotential Line. A line along which the velocity potential ϕ is constant, is called equipotential line.

For equipotential line ϕ = Constant

$\therefore \qquad d\psi = 0$

But $\qquad \phi = f(x,y)$ for steady flow

$$\therefore \qquad d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$$

$$= -udx - vdy \qquad \left\{\because \frac{\partial\phi}{\partial x} = -u, \frac{\partial\phi}{\partial y} - v\right\}$$

$$= -(udx + vdy)$$

For equipotential line, $d\phi$ =0

$$-(udx + vdy) = 0 \text{ or } udx + vdy = 0$$

$$\therefore \qquad \frac{dy}{dx} = -\frac{u}{v} \qquad \ldots\ldots\ldots\ldots(2)$$

But $\frac{dy}{dx}$ = slope of equipotential line.

Line of Constant Stream Function

$$\psi = \text{Constant}$$

$$\therefore \qquad d\psi = 0$$

But $d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi\phi}{\partial y}dy + vdx - udy$ $\quad (\because \frac{\partial\psi}{\partial x} = v; \frac{\partial\psi}{\partial y} = -u)$

For a line of constant stream function

$= \quad d\psi = 0$ or $vdx - udy = 0$

$$\frac{dy}{dx} = \frac{v}{u} \qquad ....(3)$$

But $\frac{dy}{dx}$ is slope of stream line.

From equation (2) and (3) it is clear that the product of the slope of the equipotential line and the slope of the stream line at the point of intersection is equal to -1. Thus the equipotential lines are orthogonal to the stream lines at all points of intersection.

Flow Net. A grid obtained by drawing a series of equipotential lines and stream lines is called a flow net. The flow net is an important tool in analyzing two-dimensional irrotational flow problems.

Relational Between Stream Function and Velocity Potential Function

From equation (5.9),

We have $u = -\frac{\partial\phi}{\partial x}$ and $v = -\frac{\partial\phi}{\partial y}$

From equation (5.12), We have $u = -\frac{\partial\psi}{\partial y}$ and $v = \frac{\partial\psi}{\partial x}$

Thus we have $u = -\frac{\partial\phi}{\partial x} \Rightarrow \frac{\partial\psi}{\partial y}$ and $v = -\frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial x}$

$$\left.\begin{aligned} \text{Hence} \quad & \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \\ \text{and} \quad & \frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial x} \end{aligned}\right\} \qquad ....(5.15)$$

The velocity potential function (ϕ) is given by an expression $\phi = -\frac{xy^3}{3} - x^2 + \frac{x^3y}{3} + y^2$

I. **Find the velocity components in x and y direction.**
II. **Show that ϕ represents a possible case of flow.**

Solution: Given: $\phi = -\frac{xy^3}{3} - x^2 + \frac{x^3y}{3} + y^2$

The partial derivatives of ϕ w.r.t. x and y are

$$\frac{\partial\phi}{\partial x} = -\frac{y^3}{3} - 2x + \frac{3x^2y}{3} \qquad ....(1)$$

and $$\frac{\partial\phi}{\partial y} = -\frac{3xy^2}{3} + \frac{x^3}{3} + 2y \qquad ....(2)$$

(i) The velocity components u and v are given by equation (5.9)

$$u = -\frac{\partial\phi}{\partial x} = -\left[-\frac{y^3}{3} - 2x + \frac{3x^2y}{3}\right] = \frac{y^3}{3} + 2x - x^2y$$

$\therefore$ $$u = \frac{y^3}{3} - 2x - x^2y$$

$\therefore$ $$v = -\frac{\partial\phi}{\partial y} = -\left[-\frac{3xy^2}{3} + \frac{x^3}{3} + 2y\right] = \frac{3xy^2}{3} - \frac{x^3}{3} - 2y = xy^2 - \frac{x^3}{3} - 2y$$

(ii) The given value of ϕ,will represent a possible case of flow if it satisfies the Laplace equation, i.e.,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

From equations (1) and (2), we have

Now $\quad \frac{\partial \phi}{\partial x} = -\frac{y^3}{3} - 2x - x^2 y$

$\therefore \quad \frac{\partial^2 \phi}{\partial x^2} = -2 + 2xy$

and $\quad \frac{\partial \phi}{\partial y} = xy^2 + \frac{x^3}{3} + 2y$

$\therefore \quad \frac{\partial^2 \phi}{\partial y^2} = -2xy + 2$

$\therefore \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = (-2+2xy)+(-2xy+2)=0$

$\therefore$ Laplace equation is satisfied and hence ϕ represent a possible case of flow.