UNIT IV

DIMENSIONAL AND MODEL ANALYSIS

INTRODUCTION

Dimensional analysis is a method of dimensions. It is a mathematical technique used in research work for design and for conducting model tests. It deals with the dimensions of the physical quantities involved in the phenomenon.

All physical quantities are measured by comparison, which is made with respect to an arbitrarily fixed value. Length L, mass M and time T are three fixed dimensions which are of importance in Fluid Mechanics.

If in any problem of fluid mechanics, heat is involved then temperature is also taken as fixed dimension. These fixed dimensions are called *fundamental dimensions or fundamental quantity.*

SECONDARY OR DERIVED QUANTITIES

Secondary or derived quantities are those quantities which possess more than one fundamental dimension. For example, velocity is denoted by distance per unit time (L/T), density by mass per unit volume (M/L^3) and acceleration by distance per second square (L/T^2).

Then velocity, density and acceleration become as secondary or derived quantities. The expressions (L/T), (M/L^3) and (L/T^2) are called the dimensions of velocity, density and acceleration respectively. The dimensions of mostly used physical quantities in Fluid Mechanics are given in Table.

S. No.	Physical Quantity	Symbol	units	Dimensions
	(a) Fundamental quantities			
1	Mass	M	kg	M
2	Length	L	m	L
3	Time	T	s	T
	(b)Geometric quantities			
4	Area	A	m^2	L^2
5	Volume	V	m^3	L^3
6	Moment of inertia	I	m^4	L^4
	(c)Kinetic quantities			
7	Velocity	v	m/s	LT^{-1}
8	Angular velocity	ω	rad/sec	T^{-1}
9	Acceleration	a	m/s^2	LT^{-2}

10	Angular acceleration	α	rad/sec^2	T^{-2}
11	Gravity	g	m/s^2	LT^{-2}
12	Discharge	Q	m^3/s	L^3T^{-1}
13	Kinematic viscosity	υ	m^2/s	L^2T^{-1}
	(d)Dynamic quantities			
14	Force	F	N (kg.m/s^2)	MLT^{-2}
15	Weight	W	N (kg.m/s^2)	MLT^{-2}
16	Specific weight	w	N/m^3	$ML^{-2}T^{-2}$
17	Density	ρ	kg/m^3	ML^{-3}
18	Dynamic viscosity	μ	N-s/m^2	$ML^{-1}T^{-1}$
19	Pressure Intensity	p	N/m^2	$ML^{-1}T^{-2}$
20	Modulus Of Elasticity	K or E		$ML^{-1}T^{-2}$
21	Work	W	N-m or J	ML^2T^{-2}
22	Energy	E	N-m or J	ML^2T^{-2}
23	Power	P	watts	Ml^2T^{-2}
24	Torque	T	N-m or J	ML^2T^{-2}
25	Momentum	M	kg-m/s	MLT^{-1}
26	Surface tension	σ	N/m	ML^{-2}
27	Shear stress	τ		$ML^{-1}T^{-2}$

DIMENSIONAL HOMOGENEITY

The law of Fourier principle of dimensional homogeneity states "an equation which expresses a physical phenomenon of fluid flow should be algebraically correct and dimensionally homogeneous".

Dimensionally homogeneous means, the dimensions of the terms of left hand side should be same as the dimensions of the terms on right hand side.

Let us consider the equation, $V=\sqrt{2gh}$

Dimension of L.H.S, $V=\frac{L}{T} = LT^{-1}$

Dimension of R.H.S, $\sqrt{2gh} = \sqrt{\frac{L}{T2}} \times L = \sqrt{\frac{L2}{T2}} = \frac{L}{T} = LT^{-1}$

Dimension of L.H.S = Dimension of R.H.S = LT^{-1}

Therefore equation,

$V=\sqrt{2gh}$ is dimensionally homogeneous. So it can be used in any system of units.

Uses of Dimensional Homogeneity

- To check the dimensional homogeneity of the given equation.
- To determine the dimension of a physical variable.
- To convert units from one system to another through dimensional homogeneity.
- It is a step towards dimensional analysis

Points to Be Remembered While Deriving Expressions Using Dimensional Analysis

1. First, the variables controlling the phenomenon should be identified and expressed in terms of primary dimensions.
2. Any mathematical equation should be dimensionally homogeneous.
3. In typical cases, a suitable mathematical model is constructed to simplify the problem with suitable assumptions.

METHODS OF DIMENSIONAL ANALYSIS

There are two methods of dimensional analysis used.

(i) Rayleigh's method

(ii) Buckingham π Theorem

RAYLEIGH'S METHOD

In this method, the expression is determined for a variable depending upon maximum three or four variables only. If the number of independent variables becomes more than four, it is very difficult to find the expression for the dependent variable. So, a functional relationship between variables is expressed in exponential form of equations.

Steps involved in Rayleigh's method

1. First, the functional relationship is written with the given data. ,

   Consider X as a variable which depends on $X_1, X_2, X_3, \dots Xn$

   So, the functional equation is written

   $$X=f(X_1, X_2, X_3, \dots Xn)$$

2. Then the equation is expressed in terms of a constant with exponents like powers of *a, b, c* ...

   Therefore, the equation is again written as

$X=\phi(X_1^a, X_2^b, X_3^c, \ldots X_n^z)$

Here, (ϕ)= Constant

$a, b, c, \ldots z$ = Arbitrary powers

3. The values of ***a, b,* c, ... *z*** are determined with the help of dimensional homogeneity. It means, the powers of the fundamental dimensions on both sides are compared to obtain the values of exponents.
4. Finally, these exponents/power values are substituted in the functional equation and simplified to obtain the suitable form.

BUCKINGHAM Π THEOREM

Rayleigh method is not helpful when the number of independent variables is more than three or four. This difficulty is eliminated in Buckingham π Theorem

It states that if there are 'n' variables in a dimensionally homogeneous equation and if these variables contain 'm' fundamental dimensions (M, L, T), then they are grouped into (n - m), dimensionless independent π-terms.

Let $X_1, X_2, X_3, \ldots, X_n$ are the variables involved in a physical phenomenon. Let X_1 be the variables involved in a physical phenomenon. Let X_1 be the dependent variables and $X_2, X_3, \ldots X_n$ are the independent variables on which X_1 depends. Then X_1 is a function of $X_2, X_3, \ldots X_n$ and mathematically, it is expressed as

$$X_1=f(X_2, X_3, \ldots X_n) \quad \ldots\ldots \quad (1)$$

Equation (1) can also be written as

$$F_1(X_1, X_2, X_3, \ldots X_n)=0 \quad \ldots\ldots \quad (2)$$

This equation is a dimensionally homogeneous equation. It contains *n* variables. If there are *'m'* fundamental dimensions then according to Buckingham-π-theorem, equation (2) can be written interms in which number of π -terms is equal to (n - *m*). Hence, equation (2) becomes

$$F(\pi_1, \pi_2, \ldots \pi_{n-m}) = 0 \quad \ldots\ldots \quad (3)$$

Each of π terms is dimensionless and independent of the system. Division or multiplication by a constant does not change the character of the π term. Each of π term contains (m +1) variables, where *m* is the number of Fundamental dimensions and is also called *repeating variables.* Let *'m'* in the above case X_2, X_3 and X_4 are repeating variables, if the fundamental dimensions (M, L, *T*) = 3 then each π term is written as

$$\pi_1 = X_2^{a1}, X_3^{b1}, X_4^{c1} . X^1$$

$$\pi_1 = X_2^{a2}, X_3^{b2}, X_4^{c2} . X^5$$

……………….

……………….

$$\pi_{n-m} = X_2^{a_{n-m}}, X_3^{b_{n-m}}, X_4^{c_{n-m}} . X^n \quad ……… \quad (4)$$

Each equation is solved by the principle of dimensional homogeneity and values of a_1, b_1, c_1 etc. are obtained. These values are substituted in equation (4) and values of $\pi_1, \pi_2, \pi_3 ......\pi_{n-m}$ are obtained. These values of **π**'s are substituted in equation (3). The final equation for the phenomenon is obtained by expressing any one of the π-terms as a function of others as

$$\Pi_1 = \varphi [\pi_2, \pi_3, ... \pi_{n-m}]$$

$$\Pi_2 = \varphi [\pi_1, \pi_3, ... \pi_{n-m}]$$

SELECTION OF REPEATING VARIABLES

There is no separate rule for selecting repeating variables. Hut the number of repeating variables is equal to the fundamental dimensions of the problem. Generally, ρ, υ, l or ρ, υ, D) are choosen as repeating variables.

It means, one refers to fluid property (ρ), one refers to flow property (υ) and the other one refers to geometric property (l or D). In addition to this, the following points should be kept in mind while selecting the repeating variables:

1. No variables should be dimensionless.
2. The selected two repeating variables should not have the same dimensions.
3. The selected repeating variables should be independent as far as possible.

STEPS TO BE FOLLOWED IN BUCKINGHAM Π METHOD

1. First, the variables involved in a given analysis are listed to study about given phenomenon thoroughly.
2. Then, these variables are expressed in terms of primary dimensions.
3. Next, the repeating variables are chosen according to the hint given in selection of repeating variables. Once, the repeating variables should be checked either those are independent or dependent variables because all should be independent variables.
4. Then the dimensionless parameters are obtained by adding one at a time repeating variables.
5. The number of π-terms involved in dimensional analysis is calculated using, n - m = Number of π terms.

   Where, n = Total number of variables involved in given analysis.

m = Number of fundamental variables.

6. Finally, each equation in exponential form is solved which means the coefficients of exponents are found by comparing both sides exponents. Then these dimensionless parameters are recombined and arranged suitably.

In most of the fluid mechanics problems, the choice of repeating variables may be (i) d, v, ρ (ii) l, v, ρ (iii) l, v, μ or (iv) d, v, μ

The efficiency η of a fan depends on density ρ, dynamic viscosity μ of the fluid, angular velocity ω, diameter D of the rotor and the discharge Q. Express η in terms of dimensionless parameters.

Solution:

η is a function of ρ, μ,ω,D and Q

$\eta = f(\rho, \mu, \omega, D, Q)$ or $f_1(\eta, \rho, \mu, \omega, D, Q) = 0$ **........ (i)**

Hence total number of variables, n = 6.

The value of m, i.e., number of fundamental dimensions for the problem is obtained by writing dimensions of each variable.

Dimensions of each variable are,

η = Dimensionless,

[$\rho = ML^{-3}$, $\mu = ML^{-1}T^{-1}$, $\omega = T^{-1}$, $D = L$ and $Q = L^{-3}T^{-1}$]

Number of π-terms = n - m = 6 - 3 = 3

Equation (i) is written as $(\pi_1, \pi_2, \pi_3) = 0$ **....... (ii)**

Each π -term contains *(m* + 1) variables, where *m* is equal to three and is also repeating variable. Choosing D, ω and ρ as repeating variables, we have

$\pi_1 = D^{a1} . \omega^{b1} . \rho^{c1} . \eta$

$\pi_2 = D^{a2} . \omega^{b2} . \rho^{c2} . \mu$

$\pi_3 = D^{a3} . \omega^{b3} . \rho^{c3} . Q$

First π – Term

$$\pi_1 = D^{a1} . \omega^{b1} . \rho^{c1} . \eta$$

Substituting dimensions on both sides of π_1,

$$M^0 L^0 T^0 = L^{a1} . (T^{-1})^{b1} . (ML^{-3})^{c1} . M^0 L^0 T^0$$

Equating the powers of M, L, T on both sides

Power of *M*, $0 = c_1 + 0$, $\mathbf{c_1 = 0}$

Power of *L*, $0 = a_1 + 0$, $\mathbf{a_1 = 0}$

Power of T, $0 = -b_1 + 0$, $\mathbf{b_1 = 0}$

Substituting the values of a_1, b_1 and c_1 in π_1, we get

$$\pi_1 = D^0 . \omega^0 . \rho^0 . \eta = \eta$$

[If a variable is dimensionless, it itself is a π - term. Here the variable η is a dimensionless and hence η is a π- term. As it exists in first π- term and hence $\pi_1 = \eta$ .Then there is no need of equating the powers. Directly the value can be obtained.]

Second π- Term

$$\pi_2 = D^{a2} . \omega^{b2} . \rho^{c2} . \mu$$

Substituting dimensions on both sides of π_2,

$$M^0 L^0 T^0 = L^{a2} . (T^{-1})^{b2} . (ML^{-3})^{c2} . M L^{-1} T^{-1}$$

Equating the powers of M, L, T on both sides

Power of M, $0 = c_2 + 1$, $c_1 = -1$ $\mathbf{c_1 = -1}$

Power of L, $0 = a_2 - 3c_2 - 1$, $a_2 = 3c_2 + 1 = -3+1 = -2$ $\mathbf{a_2 = -2}$

Power of T, $0 = -b_2 + -1$, $b_1 = -1$ $\mathbf{b_1 = -1}$

Substituting the values of a_2, b_2 and c_2 in π_2, we get

$$\pi_2 = D^{-2} . \omega^{-1} . \rho^{-1} . \mu = \frac{\mu}{D^2 \omega \rho}$$

Third π- Term

$$\pi_3 = D^{a3} . \omega^{b3} . \rho^{c3} . Q$$

Substituting dimensions on both sides of π_3,

$$M^0 L^0 T^0 = L^{a3} . (T^{-1})^{b3} . (ML^{-3})^{c3} . L^{-3} T^{-1}$$

Equating the powers of M, L, T on both sides

Power of M, $0 = c_3$, $c_3 = 0$ $\mathbf{c_3 = 0}$

Power of L, $0 = a_3 - 3c_3 + 3$, $a_3 = 3c_3 - 3 = -3$ $\mathbf{a_3 = -3}$

Power of T, $0 = -b_3 - 1$, $b_3 = -1$ $\mathbf{b_3 = -1}$

Substituting the values of a_2, b_2 and c_2 in π_2, we get

$$\pi_3 = D^{-3} . \omega^{-1} . \rho^0 . Q = \frac{Q}{D^2 \omega}$$

Substituting the values of π_1, π_2 and π_3 in equation (ii), we get

$$f_1\left(\eta , \frac{\mu}{D^2 \omega \rho} , \frac{Q}{D^2 \omega}\right) = 0$$

$$\eta = \Phi\left(\frac{\mu}{D^2 \omega \rho} , \frac{Q}{D^2 \omega}\right)$$ **Ans**