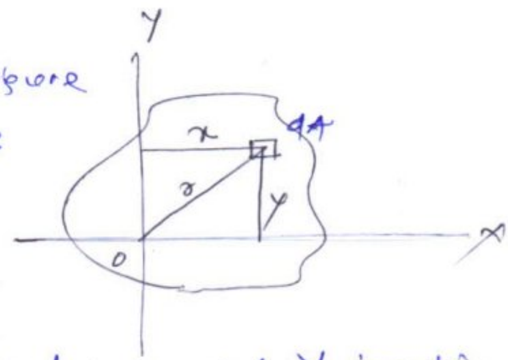
Moment of Inertia of Plane figures

02/12/14 ①

The moment of inertia of any plane figure with respect to x and y axes in its plane are expressed as



$$I_x = \int y^2 \, dA \qquad I_y = \int x^2 \, dA$$

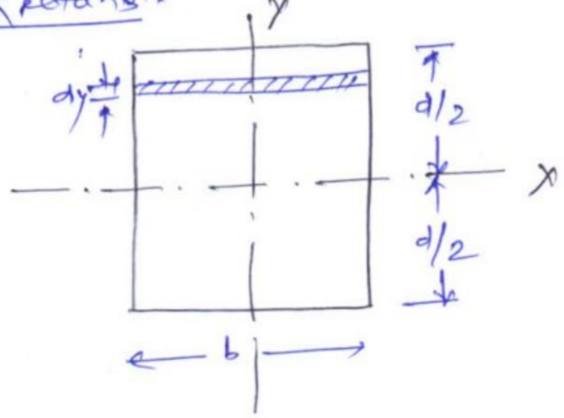
- I_{xx} and I_{yy} are also known as second moment of ~~inertia~~ area about the axes as it is distance is squared from corresponding axis.

Unit

Unit of moment of inertia of area is expressed as m^4 or mm^4.

Moment of inertia of Plane figures:-

(i) Rectangle



Considering a rectangle of width b and depth d.
Moment of inertia about centroidal axis x-x parallel to the short side i.e b
Now considering an elementary strip of width dy

Moment of inertia of the elemental strip about centroidal axis xx is

$$I_{xx} = y^2 \, dA$$
$$= y^2 b \, dy$$

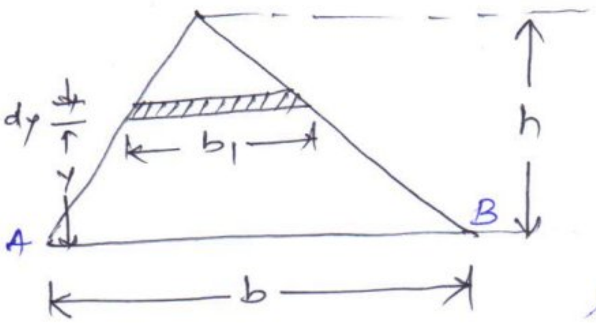
So moment of inertia of entire ~~figure~~ area

$$I_{xx} = \int_{-d/2}^{d/2} y^2 b \, dy = b \left| \frac{y^3}{3} \right|_{-\frac{d}{2}}^{d/2} = b\left[\frac{d^3}{24} + \frac{d^3}{24}\right]$$

$$\Rightarrow \boxed{I_{xx} = \frac{bd^3}{12}}$$

Similarly ~~moment at~~ $\boxed{I_{yy} = \frac{db^3}{12}}$

(ii) Triangle :- (Moment of inertia of a triangle about it's b



Consider a small elementary strip at a distance y from the base of thickness dy. Let dA is the area of strip

$dA = b_1 dy$

And $b_1 = \frac{(h-y)}{h} \times b$.

Moment of inertia of strip about base AB

$= y^2 dA = y^2 b_1 dy$

$= y^2 \frac{(h-y)}{h} . b\, dy$

$\therefore$ Moment of inertia of the triangle about AB

$$I_{AB} = \int_0^h \frac{y^2(h-y)b\,dy}{h} = \int_0^h \left(y^2 - \frac{y^3}{h}\right) b\,dy$$

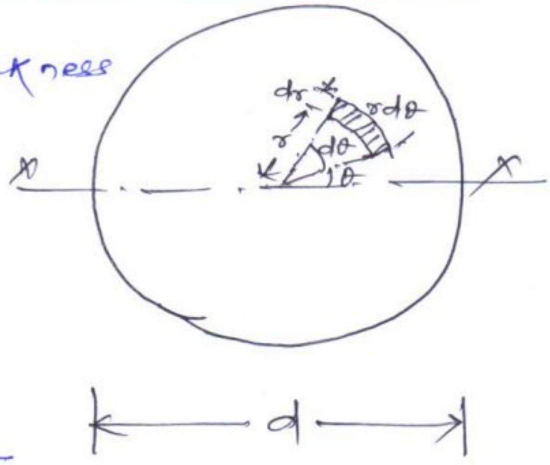
$$= b\left[\frac{y^3}{3} - \frac{y^4}{4h}\right]_0^h = b\left[\frac{h^3}{3} - \frac{h^4}{4h}\right]$$

$$= b\left[\frac{h^3}{3} - \frac{h^3}{4}\right] = \frac{bh^3}{12}$$

$$\Rightarrow \boxed{I_{AB} = \frac{bh^3}{12}}$$

(iii) Moment of inertia of a circle about it's centroidal axis

Considering an elementary strip of thickness dr, the side of strip $r\,d\theta$.



moment of inertia of strip about xx

$= y^2 dA$

$= (r \sin\theta)^2\, r\,d\theta\, dr$

$= r^3 \sin^2\theta\, d\theta\, dr$

$\therefore$ Moment of inertia of circle about xx axis

$$I_{xx} = \int_0^R \int_0^{2\pi} r^3 \sin^2\theta\, d\theta\, dr$$

$$= \int_0^R \int_0^{2\pi} r^3 \left(\frac{1-\cos 2\theta}{2}\right) d\theta\, dr$$

$$= \int_0^R \frac{r^3}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} dr$$

$$= \int_0^R \frac{r^3}{2}\left(2\pi - \frac{\sin 4\pi}{2}\right) dr$$

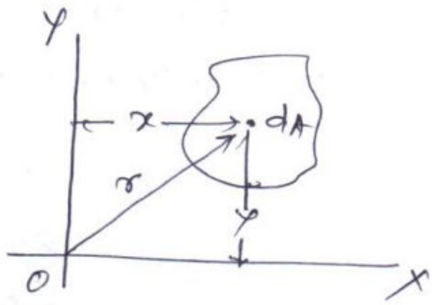
$$= \left[\frac{r^4}{8}\right]_0^R \left[2\pi - 0\right]$$

$$= \frac{R^4}{8} 2\pi = \frac{\pi R^4}{4}$$

$$\Rightarrow \boxed{I_{xx} = \frac{\pi R^4}{4} = \frac{\pi D^4}{64}} \quad \left(\because R = \frac{D}{2}\right)$$

Polar moment of inertia :-

Moment of inertia about an axis perpendicular to the plane of area is called polar moment of inertia it may denoted as J or I_{zz}



$$\boxed{I_{zz} = \Sigma r^2 dA}$$

Radius of Gyration :-

Radius of gyration may be defined by a relation

$$\boxed{K = \sqrt{\frac{I}{A}}}$$

where K = radius of gyration
I = moment of inertia
A = cross-sectional area

so, we can have the following relations

$$\boxed{\begin{aligned} K_{xx} &= \sqrt{\frac{I_{xx}}{A}} \\ K_{yy} &= \sqrt{\frac{I_{yy}}{A}} \\ K_{AB} &= \sqrt{\frac{I_{AB}}{A}} \end{aligned}}$$

Theorems of Moment of inertia

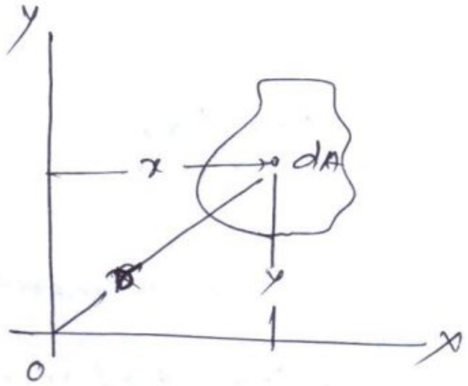
There are two theorems of moment of inertia

(a) perpendicular axis theorem

(b) parallel axis theorem.

Perpendicular axis theorem:-

Moment of inertia of an area about an axis ⊥r to it's plane at any point O is equal to the sum of moments of inertia about any two mutually perpendicular axis through the same point O and lying in the plane of area.



$$I_{zz} = I_{xx} + I_{yy}$$

$$I_{zz} = \Sigma r^2 dA$$

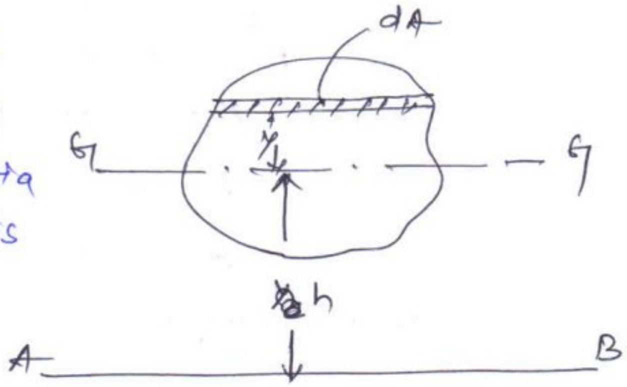
$$= \Sigma (x^2 + y^2)\, dA$$

$$= \Sigma x^2 dA + \Sigma y^2 dA$$

$$\Rightarrow \boxed{I_{zz} = I_{xx} + I_{yy}}$$

Parallel axis theorem:-

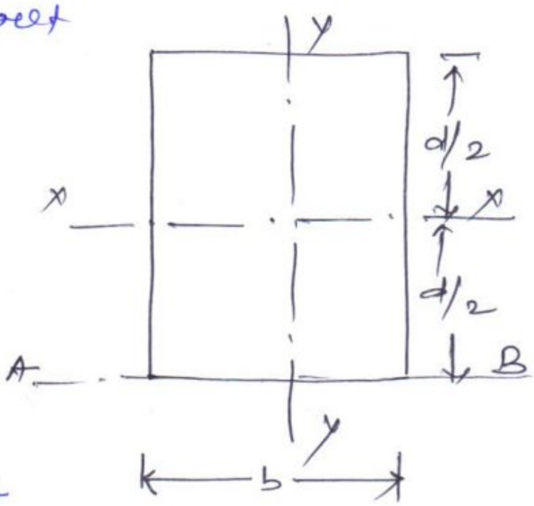
Moment of inertia about an axis in the plane of an area is equal to the sum of moment of inertia about a parallel centroidal axis and the product of area and square of the distance betn the two parallel axes.



$$\boxed{I_{AB} = \cancel{I_{xx}}\ I_{GG} + Ah^2}$$

Moment of inertia of standard Sections:-

i) Moment of inertia of a rectangle about its centroidal axis xx



$$I_{xx} = \frac{bd^3}{12}$$

Similarly moment of inertia about it's centroidal axis yy

$$I_{yy} = \frac{db^3}{12}$$

Now moment of inertia of rectangle about it's base AB can be obtained by applying parallel axis theorem

$$I_{AB} = I_{xx} + Ah^2$$

$$= \frac{bd^3}{12} + (bd)\left(\frac{d}{2}\right)^2$$

$$= \frac{bd^3}{12} + \frac{bd^3}{4}$$

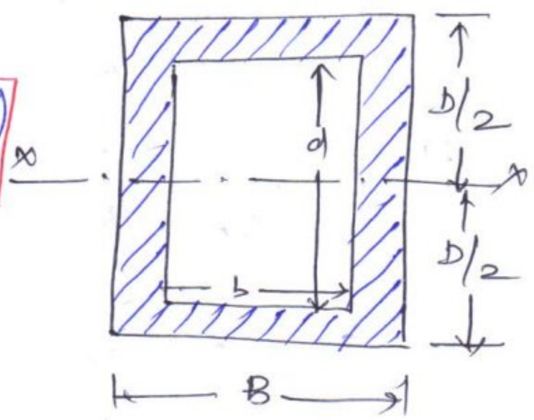
$$= \frac{3bd^3 + bd^3}{12} = \frac{bd^3}{3}$$

$$\Rightarrow \boxed{I_{AB} = \frac{bd^3}{3}}$$

(ii) Moment of inertia of a hollow rectangular section:-

Moment of inertia of hollow rectangular section

$$\boxed{I_{xx} = \frac{BD^3}{12} - \frac{bd^3}{12} = \frac{1}{12}\left(BD^3 - bd^3\right)}$$



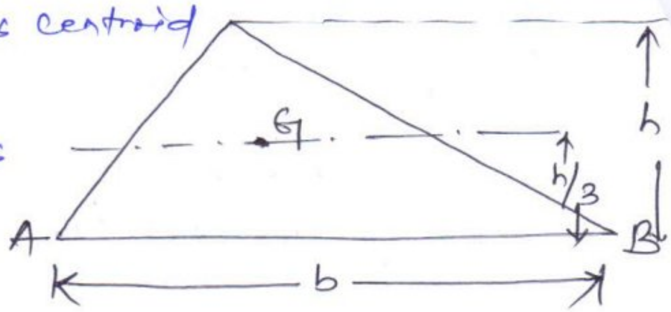
(iii) Moment of inertia of triangle about it's ~~base~~ centroidal

Moment of inertia of triangle about it's base

= moment of inertia about its centroid

+ Ah^2

(using parallel axis theorem)



$$\Rightarrow I_{AB} = I_{xx} + Ah^2$$

$$\Rightarrow \frac{bh^3}{12} = I_{xx} + \frac{1}{2} b \times h \times \left(\frac{h}{3}\right)^2$$

$$= I_{xx} + \cancel{\frac{bh^3}{2}} \frac{bh^3}{18}$$

$$\Rightarrow I_{xx} = \frac{bh^3}{12} - \frac{bh^3}{18} = \cancel{\frac{6bh^3 - bh^3}{12}}$$

$$= \frac{3bh^3 - 2bh^3}{36} = \frac{bh^3}{36}$$

$$\Rightarrow \boxed{I_{xx} = \frac{bh^3}{36}}$$

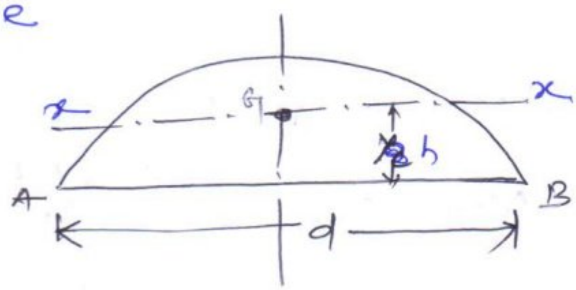
(iv) Moment of inertia of semicircle

(a) about diametral axis

Moment of inertia of semicircle

about $AB = \frac{1}{2} \frac{\pi d^4}{64}$

$$= \boxed{\frac{\pi d^4}{128}}$$



(b) about centroidal axis xx

$$\boxed{h = \frac{4R}{3\pi} = \frac{2d}{3\pi}}$$

area $A = \frac{1}{2} \frac{\pi d^2}{4} = \frac{\pi d^2}{8}$

Using parallel axis theorem

$$I_{AB} = I_{xx} + Ah^2$$

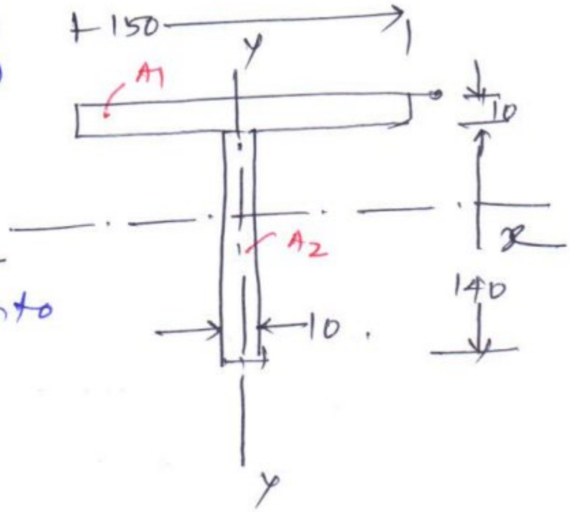
$$\Rightarrow \frac{\pi d^4}{128} = I_{xx} + \frac{\pi d^2}{8} \times \left(\frac{2d}{3\pi}\right)^2 = \cancel{I_{xx} + \frac{\pi d^3}{12\pi}}$$

$$\Rightarrow \frac{\pi d^4}{128} = I_{xx} + \frac{\pi d^2}{8} \times \frac{4d^2}{9\pi^2}$$

$$= I_{xx} + \frac{d^4}{18\pi}$$

$$\Rightarrow \boxed{I_{xx} = \left(\frac{\pi d^4}{128} - \frac{d^4}{18\pi}\right)}$$

Moment of inertia of composite figures :-



Q.1 Determine the moment of inertia of the composite section about an axis passing through the centroidal axis. Also determine MI about axis of symmetry and radius of gyration.

Solⁿ Dividing the composite area into A_1 and A_2

$A_1 = 150 \times 10 = 1500 \text{ mm}^2$

$A_2 = 140 \times 10 = 1400 \text{ mm}^2$

Distance of centroid from base of the composite figure

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{(A_1 + A_2)} = \frac{1500 \times 145 + 1400 \times 70}{2900}$$

$$= 108.79 \text{ mm}$$

Moment of inertia of the area about xx axis

$$I_{xx} = \left\{\frac{150 \times 10^3}{12} + 1500 \times (145 - 108.79)^2\right\} + \left\{\frac{10 \times 140^3}{12} + 1400 \times (108.79 - 70)^2\right\}$$

$$= (12500 + 1966746.15) + (2286666.667 + 2106529.74)$$

$$= 6372442.557 \text{ mm}^4$$

Similarly

$$I_{yy} = \frac{10 \times 150^3}{12} + \frac{140 \times 10^3}{12} = 2812500 + 11666.66667$$

$$= 2824166.667 \text{ mm}^4$$

Radius of gyration $k = \sqrt{\frac{I}{A}}$

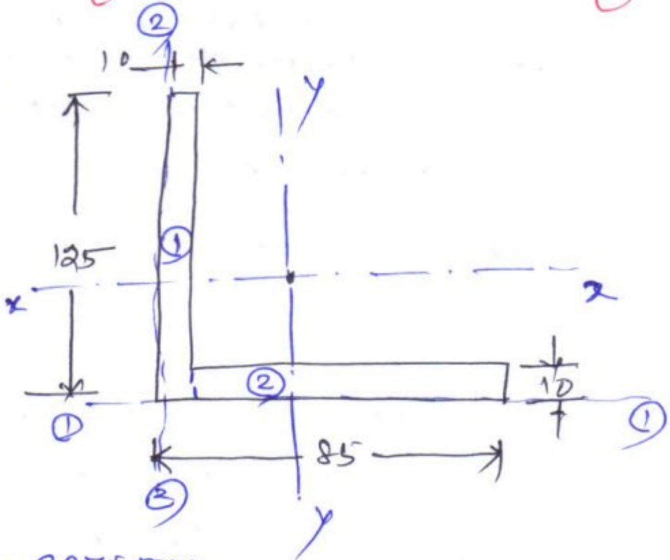
so $K_{xx} = \sqrt{\frac{I_{xx}}{A}}$

$$= \sqrt{\frac{6372442.5}{2900}} = 46.87 \text{ mm}$$

Similarly $k_{yy} = \sqrt{\frac{I_{yy}}{A}} = \sqrt{\frac{2824166.667}{2900}}$

$= 31.206$ mm (Ans)

Q:2 Determine the MI of L-section about its centroidal axes parallel to the legs. Also find the polar moment of inertia.



We have $A_1 = 125 \times 10 = 1250 \text{ mm}^2$

$A_2 = 75 \times 10 = 750 \text{ mm}^2$

Total area $A_1 + A_2 = 2000 \text{ mm}^2$

Distance of centroid from 1-1 axis

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2}$$

$$= \frac{1250 \times 62.5 + 750 \times 5}{2000} = 40.9375 \text{ mm}$$

Distance of centroidal axis yy from 2-2 axis

$$\bar{x} = \frac{A_1 x_1 + A_2 x_2}{A_1 + A_2}$$

$$= \frac{1250 \times 5 + 750 \times \left(\frac{75}{2} + 10\right)}{2000}$$

$$= \frac{1250 \times 5 + 750 \times 47.5}{2000} = 20.93 \text{ mm}$$

Moment of inertia about xx axis

$$I_{xx} = \left\{\frac{10 \times 125^3}{12} + 1250 \times (62.5 - 40.9375)^2\right\}$$

$$+ \left\{\frac{75 \times 10^3}{12} + 750 \times (40.9375 - 5)^2\right\}$$

$$= (1627604.167 + 581176.7578) + (6250 + 968627.9297)$$

$$= \boxed{3183658.854 \text{ mm}^4}$$

Similarly MI about yy centroidal axis

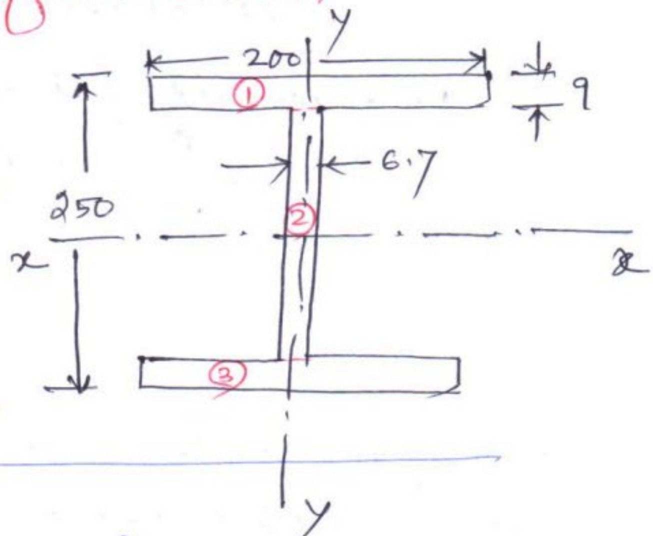
$$I_{yy} = \left\{\frac{125\times10^3}{12} + 1250\times(20.93-5)^2\right\}$$
$$+ \left\{\frac{10\times75^3}{12} + 750\times(47.5-20.93)^2\right\}$$
$$= (10416.66667 + 317206.125) + (351562.5 + 529473.675)$$
$$= \boxed{1208658.967\ mm^4}$$

Polar moment of inertia $I_{zz} = I_{xx} + I_{yy}$

$$= \boxed{4392317.821\ mm^4} \quad (Ans)$$

Q.3 Determine the MI of the symmetrical I section about its centroidal axes x-x and y-y. Also determine the polar moment of inertia of the section.

We have from the figure



$A_1 = 200\times9 = 1800\ mm^2$

$A_2 = 232\times6.7 = 1554.4\ mm^2$

$A_3 = 200\times9 = 1800\ mm^2$

~~Position of centroidal axis x x from base~~

$$\bar{y} = \frac{A_1y_1 + A_2y_2 + A_3y_3}{(A_1+A_2+A_3)}$$

$$= \frac{1800\times(4.5+232+9) + 1554.4\times\left(\frac{232}{2}+9\right) + 1800\times4.5}{(1800+1554.4+1800)}$$

$$= \frac{1800\times245.5 + 1554.4\times125 + 1800\times4.5}{(1800+1554.4+1800)}$$

$$= 125\ mm.$$

$$\bar{x} = \frac{A_1x_1 + A_2x_2 + A_3x_3}{(A_1+A_2+A_3)}$$

$$= \frac{1800\times100 + 1554.4\times96.65 + 1800\times100}{(1800+1554.4+1800)} = 98.98$$

MI about xx axis

$$I_{xx} = \left\{\frac{200\times 9^3}{12} + 1800\times(125-4.5)^2\right\} + \left\{\frac{6.7\times 232^3}{12} + 1554.4\times(\right.$$

$$+ \left\{\frac{200\times 9^3}{12} + 1800\times(125-4.5)^2\right\}$$

$$= (12150 + 26136450) + (6972002.133 + 0)$$

$$+ (12150 + 26136450)$$

$$= 26148600 + 6972002.133 + 26148600$$

$$= \boxed{59269202.13 \text{ mm}^4}$$

MI about yy axis

$$I_{yy} = \frac{9\times 200^3}{12} + \frac{232\times 6.7^3}{12} + \frac{9\times 200^3}{12}$$

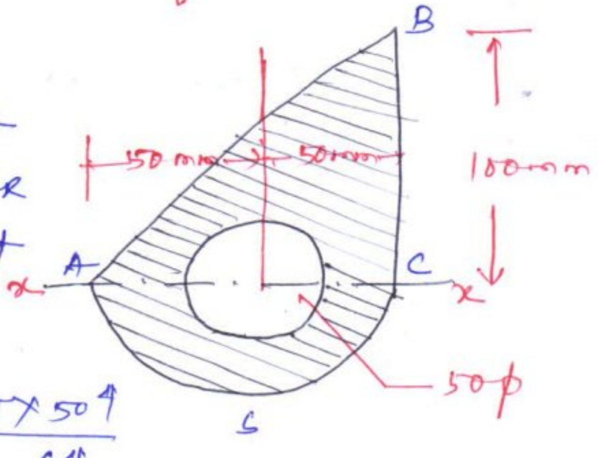
$$= 6000000 + 5814.751 + 6000000$$

$$= \boxed{12005814.75 \text{ mm}^4}$$

Polar moment of inertia $I_{zz} = I_{xx} + I_{yy}$

$$= \boxed{71275016.88 \text{ mm}^4}$$

Q.4 Calculate the moment of inertia of the shaded area about xx axis.



MI of the shaded section about xx = MI of triangle ABC about xx + MI of semicircle ACS about xx − MI of circle

$$= \frac{100\times 100^3}{12} + \frac{\pi\times 100^4}{128} - \frac{\pi\times 50^4}{64}$$

$$= 8333333.333 + 2454369.261 - 306796.1576$$

$$= 10480906.44 \text{ mm}^4$$

$$= \boxed{1.048\times 10^7 \text{ mm}^4}$$