Unit- 1

Functions of Complex variable

Definitions:

1. Complex Variables:- If $z = x+iy$, then z is called a complex variable. Also x and y are called real and imaginary parts of z.

For example:- $z = -3-5i$.

2. Functions of a complex variable.

Let $w = u+iv$ and $z = x+iy$ be two complex variables. w is said to be a function of z, if to every value of z in some domain (region) D, there correspond one or more than one values of w. It is written as :

$$w = f(z) = u+iv$$

or

$$w = u(x,y)+iv(x,y)$$

For example: 1. $f(z) = z^2$ 2. $f(z) = \sin z$,

3. Single valued and multi-valued functions.

If w takes only one value for each value of z in domain D, then w is called single valued function of z. If w takes more than one values for some or all values of z in D, then w is said to be multi-valued (or many valued) function of z.

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4. **Analytic Function.** A single valued function $f(z)$ in a domain D is said to be analytic at a point $z = a$ if there exists a neighbourhood $|z - a| < \delta$ at all points of which $f'(z)$ exists.

5. **Cauchy - Riemann Equations. (C-R Equations).**

$f(z) = u(x,y) + i v(x,y)$ be analytic in a domain D if u and v satisfy Cauchy - Riemann equations

$$\boxed{\begin{gathered}\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \\ \text{or} \\ u_x = v_y \quad \text{and} \quad u_y = -v_x.\end{gathered}}$$

6. **Necessary condition for $f(z)$ to be analytic:**

If $f(z) = u + iv$ is analytic in a domain D, then u, v satisfy the C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{must hold.}$$

7. **Harmonic Function.**

A real valued function $u = u(x,y)$ is called harmonic function if,

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

8. **Harmonic Conjugate Functions.**

If the function $f(z) = u(x,y) + i v(x,y)$ is analytic in a domain D, then the functions u and v are said to conjugate functions.

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Q.1 If $f(z)$ is a regular function of z, prove that
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2.$$

Sol. We know that:
$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = 4\cdot\frac{\partial^2}{\partial z\,\partial \bar{z}}$$

where $z = x+iy$ & $\bar{z} = x-iy$.

$\therefore$ L.H.S. $= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2$

$= 4\cdot\frac{\partial^2}{\partial z\partial\bar{z}}|f(z)|^2$

$= 4\cdot\frac{\partial^2}{\partial z\partial\bar{z}}(f(z)\cdot f(\bar{z}))$ $[\because |z|^2 = z\bar{z}]$

$= 4\cdot\frac{\partial}{\partial z}\left[\frac{\partial}{\partial\bar{z}}f(z)f(\bar{z})\right]$

$= 4\cdot\frac{\partial}{\partial z}\left[f(z)\cdot f'(\bar{z})\right]$

$= 4\cdot f'(z)\cdot f'(\bar{z}) = 4|f'(z)|^2$ $[\because |z|^2 = z\bar{z}]$

$=$ R.H.S.

Q.2. If $f(z)$ is a regular function of z, prove that
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$$

Sol. We know that:
$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\partial\bar{z}}$$

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$\therefore$ L.H.S $= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log |f'(z)|$

$= 4 \cdot \frac{\partial^2}{\partial z \partial \bar{z}} \left\{ \frac{1}{2} \log |f'(z)|^2 \right\}$

$= 2 \cdot \frac{\partial^2}{\partial z \partial \bar{z}} \left[\log \{ f'(z) \, \bar{f}'(\bar{z}) \} \right] \quad [\because |z|^2 = z\bar{z}]$

$= 2 \cdot \frac{\partial^2}{\partial z \partial \bar{z}} \left[\log f'(z) + \log \bar{f}'(\bar{z}) \right]$

$= 2 \frac{\partial}{\partial z} \left[\frac{\bar{f}''(\bar{z})}{\bar{f}'(\bar{z})} \right]$

$= 0 \qquad [\because \bar{f}'(\bar{z}) \ \& \ \bar{f}''(\bar{z})$ are constant w.r.t z.$]$

$=$ R.H.S.

Q.3. Determine whether $\frac{1}{z}$ is analytic or not?

sol. Let $f(z) = u + iv$

$\Rightarrow u + iv = \frac{1}{z} \qquad [\because f(z) = \frac{1}{z}]$

$\Rightarrow u + iv = \frac{1}{x + iy} \qquad [\because z = x + iy]$

$\Rightarrow u + iv = \frac{x - iy}{(x+iy)(x-iy)} = \frac{x - iy}{x^2 + y^2}$

Equating real and Imaginary parts, we get

$u = \frac{x}{x^2 + y^2}, \quad v = \frac{-y}{x^2 + y^2}$

$\therefore \frac{\partial u}{\partial x} = \frac{(x^2+y^2) \cdot 1 - x \cdot 2x}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$

and $\frac{\partial u}{\partial y} = \frac{(x^2+y^2) \cdot 0 - x \cdot 2y}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$

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$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2)\cdot 0 + y\cdot(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

& $$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)(-1) + y\,(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

clearly, $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{-2xy}{(x^2+y^2)^2}.$$

Thus C-R equations are satisfied. Therefore $\frac{1}{z}$ is analytic.

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Unit - 4

Functions of Complex Variable

Method for constructing an analytic function.

Method I.

case. I If u is given function then to find v:

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

we know that C-R equations is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$dv = Mdx + Ndy \quad \text{---} \ (1)$$

where $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$

Differentiate, $\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$

Since u is harmonic $\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad [\because \text{By C-R equations}]$$

Hence equ. (1) is exact differential equation.

$\therefore$ dv can be integrated to get v.

i.e. $\int dv = \int M dx + \int N dy + c$

(free from x)

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Case II: If v is given function then to find u:

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

we know that C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore du = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$$

$$du = M dx + N dy \quad \text{—— (1)}$$

where $M = \frac{\partial v}{\partial y}$ and $N = -\frac{\partial v}{\partial x}$

Differentiate: $\frac{\partial M}{\partial y} = \frac{\partial^2 v}{\partial y^2}$ and $\frac{\partial N}{\partial x} = \frac{\partial^2 v}{\partial x^2}$

Since v is harmonic $\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad [\because \text{C-R equations}]$$

Hence equ. (1) is exact differential equation.

$\therefore$ du can be integrated to get u.

i.e. $\int du = \int M dx + \int N dy + c$

(free from x)

Q.1. Show that $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic and find its harmonic conjugate.

Sol. Here, $u = \frac{1}{2}\log(x^2+y^2)$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2+y^2} \cdot 2x = \frac{x}{x^2+y^2}$$

and $\frac{\partial u}{\partial y} = \frac{1}{2} \cdot \frac{1}{x^2+y^2} \cdot 2y = \frac{y}{x^2+y^2}$

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and $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{(x^2+y^2).1 - x.(2x)}{(x^2+y^2)^2} = \dfrac{x^2+y^2-2x^2}{(x^2+y^2)^2}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

and $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{(x^2+y^2).1 - y(2y)}{(x^2+y^2)^2} = \dfrac{x^2+y^2-2y^2}{(x^2+y^2)^2}$

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

clearly, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2} + \dfrac{x^2-y^2}{(x^2+y^2)^2}$

$$= \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2}$$

i.e $\boxed{\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u \text{ is harmonic.}}$

Let the harmonic conjugate of u be v. Then

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial y}dy \qquad [\because \text{By C-R equations}]$$

or $dv = -\dfrac{y}{x^2+y^2}dx + \dfrac{x}{x^2+y^2}dy$

$$= M\,dx + N\,dy \quad (\text{say}) \quad \text{———} (1)$$

where $M = \dfrac{-y}{(x^2+y^2)}$ & $N = \dfrac{x}{(x^2+y^2)}$

$\therefore \cancel{\dfrac{\partial M}{\partial y} = \dfrac{\partial^2 u}{\partial x^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}}$

$\cancel{\dfrac{\partial N}{\partial x} = \dfrac{\partial^2 u}{\partial y^2} = y^2-x^2}$

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$$\therefore \frac{\partial M}{\partial y} = \frac{(x^2+y^2)(-1)+y(2y)}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\& \ \frac{\partial N}{\partial x} = \frac{(x^2+y^2).1-x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

clearly, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ equ. (1) is an exact.

$\therefore$ It's solution is

$$\int dv = \int M dx + \int N dy + c$$

(free from x)

$$v = \int \frac{-y}{x^2+y^2} dx + \int 0 \, dy + c$$

$$= -y . \frac{1}{y} \tan^{-1}\left(\frac{x}{y}\right) + c$$

$$\boxed{v = -\tan^{-1}\left(\frac{x}{y}\right) + c}$$

Q.2. Show that the function $v = \sinh x . \cos y$ is harmonic and find its harmonic conjugate.

Sol. Given: $v = \sinh x . \cos y$ ——(1)

Now, $\frac{\partial v}{\partial x} = \cosh x . \cos y \Rightarrow \frac{\partial^2 v}{\partial x^2} = \sinh x . \cos y$ ——(2)

and $\frac{\partial v}{\partial y} = -\sinh x . \sin y \Rightarrow \frac{\partial^2 v}{\partial y^2} = -\sinh x . \cos y$ ——(3)

Then $\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \Rightarrow v \text{ is harmonic.}}$

we know that:

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

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$$du = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy \qquad [\because \text{By C-R equations}]$$

or $du = (-\sinh x.\sin y)dx - (\cosh x.\cos y)dy$

$$= M\,dx + N\,dy \quad \text{—— (4)}$$

Here, $M = -(\sinh x \sin y)$ & $N = -(\cosh x.\cos y)$

$\therefore \frac{\partial M}{\partial y} = -\sinh x.\cos y$ & $\frac{\partial N}{\partial x} = -\sinh x.\cos y$

clearly, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\Rightarrow$ equ. (4) is an exact.

$\therefore$ It's solution:

$$\int du = \int M\,dx + \underset{(\text{free from } x)}{\int N\,dy} + C$$

$$u = \int -(\sinh x.\sin y)dx + \int 0\,dy + c$$

$$= -\sin y.\int \sinh x\,dx + c$$

$$\boxed{u = -\sin y \cosh x + c}$$

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Unit-4

Functions of Complex Variable

Method For Constructing an analytic Function:

Method II: [Milne's Thomson Method].

Type 1. To construct analytic function $f(z)$ in terms of z, when real part is given by the following formula:

$$\boxed{f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + c}$$

where, $\phi_1(x,y) = \frac{\partial u}{\partial x}$ and $\phi_2(x,y) = \frac{\partial u}{\partial y}$

for $\phi_1(z,0)$ & $\phi_2(z,0)$ put $x = z$, & $y = 0$.

Type 2. To construct analytic function $f(z)$ in terms of z, when imaginary part v is given by the following formula,

$$\boxed{f(z) = \int [\phi_1(z,0) + i\phi_2(z,0)]\, dz + c}$$

where, $\phi_1(x,y) = \frac{\partial v}{\partial y}$ and $\phi_2(x,y) = \frac{\partial v}{\partial x}$.

for $\phi_1(z,0)$ & $\phi_2(z,0)$ put $x = z$, & $y = 0$.

Type 3. To construct analytic function $f(z)$ in terms of z, where $u - v$ is given:

Let $U = u - v$, then

$$\boxed{(1+i) f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + c}$$

where $\phi_1(x,y) = \frac{\partial U}{\partial x}$ and $\phi_2(x,y) = \frac{\partial U}{\partial y}$.

Type 4. To construct analytic function $f(z)$ in terms of z, when $u+v$ is given:

Let $V = u + v$, then

$$\boxed{(1+i) f(z) = \int [\phi_1(z,0) + i\phi_2(z,0)]\, dz + c}$$

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where $\phi_1(x,y) = \frac{\partial V}{\partial y}$ and $\phi_2(x,y) = \frac{\partial V}{\partial x}$.

Q.1. Show that the function $u = x^3 - 3xy^2$ is harmonic and find the corresponding analytic function of this as the real part.

Sol. Given: $u = x^3 - 3xy^2$

$$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2 \text{ and } \frac{\partial^2 u}{\partial x^2} = 6x$$

$$\frac{\partial u}{\partial y} = -6xy \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -6x.$$

Hence, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0 \Rightarrow u$ is harmonic.

Now, we have to find $f(z) = u + iv$

By Milne's Thomson Method.

$$f(z) = \int [\phi_1(z,0) + i\,\phi_2(z,0)]\, dz + C \quad \text{——} (1)$$

Here $\phi_1(x,y) = \frac{\partial u}{\partial x} = 3x^2 - 3y^2$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = -6xy$$

Replacing $x \to z$ and $y \to 0$, we get

$$\phi_1(z,0) = 3z^2 \;\&\; \phi_2(z,0) = 0$$

Hence (1) becomes

$$f(z) = \int [3z^2 + i.0]\, dz + iC$$

$$\boxed{f(z) = z^3 + iC}$$

Q.2. Show that $e^x(x\cos y - y\sin y)$ is harmonic function. Find the analytic function for which $e^x(x\cos y - y\sin y)$ is imaginary part.

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Sol. Given: $v = e^x(x\cos y - y\sin y)$ —— (1)

$\therefore \frac{\partial v}{\partial x} = e^x(x\cos y - y\sin y) + e^x(\cos y)$ —— (2)

and $\frac{\partial v}{\partial y} = 0 + e^x(-x\sin y - \sin y - y\cos y)$ —— (3)

Again diff. (2) and (3), we get

$$\frac{\partial^2 v}{\partial x^2} = e^x(x\cos y - y\sin y) + e^x\cos y + e^x\cos y$$

$$= e^x(x\cos y - y\sin y + 2\cos y) \quad \text{—— (4)}$$

$$\frac{\partial^2 v}{\partial y^2} = e^x(-x\cos y - \cos y - \cos y + y\sin y)$$

$$= e^x(-x\cos y - 2\cos y + y\sin y) \quad \text{—— (5)}$$

Adding equ. (4) and (5), we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \Rightarrow v \text{ is harmonic.}$$

Now, Milne Thomson method, if v is given:

$$f(z) = \int [\phi_1(z,0) + i\phi_2(z,0)]\, dz + C \quad \text{—— (6)}$$

where $\phi_2(x,y) = \frac{\partial v}{\partial x} = e^x(x\cos y - y\sin y) + e^x(\cos y)$

Then $\phi_2(z,0) = e^z(z - 0) + e^z(1) = e^z(z+1)$

and $\phi_2(x,y) = \frac{\partial v}{\partial y} = e^x(-x\sin y - \sin y - y\cos y)$

then $\phi_1(z,0) = e^z(0-0-0) = 0$

Hence (6) becomes:

$$f(z) = \int [0 + i(z+1)e^z]\, dz + C$$

$$= i\left[(z+1)e^z - \int 1.e^z dz\right] + C = i\left[(z+1)e^z - e^z\right] + C$$

$$\boxed{f(z) = ize^z + C}$$

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Q3. If $f(z) = u + iv$ is an analytic function of z, find $f(z)$, if $u - v = e^{x}(\cos y - \sin y)$.

Sol. Let $U = u - v = e^{x}(\cos y - \sin y)$ —— (1)

By Milne's - Thomson method, if $U = u - v$ is given, then

$$(1+i)f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz + C \quad \text{—— (2)}$$

where, $\phi_1(x,y) = \frac{\partial U}{\partial x} = e^{x}(\cos y - \sin y)$

and $\phi_2(x,y) = \frac{\partial U}{\partial y} = e^{x}(-\sin y - \cos y)$

Put $x \to z$ & $y \to 0$

$\phi_1(z,0) = e^{z}(1-0) = e^{z}$

$\phi_2(z,0) = e^{z}(0-1) = -e^{z}$

Hence (2) becomes :

$$(1+i)f(z) = \int [e^{z} - i(-e^{z})]\,dz + C$$

or $(1+i)f(z) = (1+i)e^{z} + C$

$\Rightarrow f(z) = e^{z} + \frac{C}{1+i}$

$$\boxed{f(z) = e^{z} + C_1}$$

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Functions of Complex Variable

Q.4. If $f(z) = u+iv$ is an analytic function of z, find $f(z)$ if
$$u+v = \frac{2\sin 2x}{(e^{2y}+e^{-2y}-2\cos 2x)}.$$

Sol. Let
$$V = u+v = \frac{2\sin 2x}{(e^{2y}+e^{-2y}-2\cos 2x)}$$

or
$$V = \frac{\sin 2x}{\left(\frac{e^{2y}+e^{-2y}}{2}\right)-\cos 2x} = \frac{\sin 2x}{\cosh 2y - \cos 2x} \quad \text{—— (1)}$$

$$\left[\because \frac{e^{2\theta}+e^{-2\theta}}{2} = \cosh 2\theta\right]$$

By Milne - Thomson method, if $V = u+v$ is given, then
$$(1+i)f(z) = \int [\phi_1(z,0)+i\,\phi_2(z,0)]\,dz + C \quad \text{—— (2)}$$

where,
$$\phi_1(x,y) = \frac{\partial V}{\partial y} = \frac{-2\sin 2x.\sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$\Rightarrow \phi_1(z,0) = 0 \qquad [\because \sinh 0 = 0]$$

and
$$\phi_2(x,y) = \frac{\partial V}{\partial x} = \frac{2\cos 2x(\cosh 2y-\cos 2x) - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$\Rightarrow \phi_2(z,0) = \frac{2\cos 2z(1-\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1-\cos 2z)^2} = \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2}$$

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$$= \frac{-2}{1-\cos 2z} = -\frac{2}{2\sin^2 z} = -\operatorname{cosec}^2 z.$$

Hence ② becomes,

$$(1+i)f(z) = \int \left[0 + i(-\operatorname{cosec}^2 z)\right] dz + c$$

$$= \cancel{\cot}\ i\cot z + c$$

or $$f(z) = \frac{i}{1+i}\cot z + \frac{c}{1+i}$$

$$= \frac{i(1-i)}{(1+i)(1-i)}\cot z + \frac{c}{1+i}$$

$$= \frac{i(1-i)}{1+1}\cot z + \frac{c}{1+i}$$

$$\boxed{f(z) = \frac{i(1-i)}{2}\cot z + c_1}$$

Cauchy's Theorem. (without proof).

Statement: If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a simple closed curve C, then

$$\int_C f(z)\,dz = 0.$$

Cauchy-Goursat's Theorem (without proof).

Statement: If a function $f(z)$ is analytic and single valued inside and on a simple closed contour C, then

$$\int_C f(z)\,dz = 0.$$

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Cauchy's Integral Formula.

Statement: If $f(z)$ is analytic within and on a closed curve C and 'a' is any point within C, then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)} dz$$

or

$$\int_C \frac{f(z)}{(z-a)} dz = 2\pi i f(a) \; ; \; a \in C.$$

In general,

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz.$$

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Unit-4

Functions of complex variables

Some Definitions:

Zero of an analytic Function: A zero of an analytic function $f(z)$ is the value of z for which $f(z)=0$.

For example: $$f(z) = \frac{z+1}{z(z-2)}$$

For zero of $f(z)$, we put $\frac{z+1}{z(z-2)} = 0$

$$\Rightarrow \quad z+1 = 0 \quad \Rightarrow \quad z = -1$$

order of zero: If, $f(z) = (z-a)^m \phi(z)$,

such that $f(a) = 0,\ f'(a) = 0,\ f''(a) = 0, \ldots, f^m(a) \neq 0.$

For example: $$f(z) = \frac{(z-i)^3}{z^2+1}$$

For zero of $f(z)$, put $f(z) = 0$, i.e.

$$\frac{(z-i)^3}{z^2+1} = 0 \quad \Rightarrow \quad (z-i)^3 = 0$$

$$\Rightarrow \quad z = i \text{ of order } m = 3.$$

Singularity of an analytic function:

A singular point of function $f(z)$ is the point at which the function not analytic. In other words a point at which function $f(z)$ is not defined.

For example: $f(z) = \frac{1}{z+2i}$ has a singularity at

$z = -2i$.

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Remark. For finding the singular point of analytic functions $f(z)$, we put denominator of $f(z) = 0$.

For example; $f(z) = \dfrac{\sin z}{(z-i)(z-\pi/2)}$

For singular points, we put:

$$(z-i)\left(z-\frac{\pi}{2}\right) = 0 \Rightarrow z = i, \pi/2, \text{ i.e}$$

i and $\pi/2$ are singular points of $f(z)$.

Types of singularity:

1. **Pole:** If the principal part of $f(z)$ at $z=a$ consists of a finite number of terms, ~~say~~ say m, then the singularity at $z=a$ is called a pole of order m of $f(z)$.

A pole of order 1 is called a simple pole.

or

If $\lim_{z \to a} f(z) = \infty$, then $z=a$ is a pole of $f(z)$.

Remark. For finding the pole of $f(z)$, we put denominator to zero.

For example: Let $f(z) = \dfrac{1}{(z-1)^3 (z-3)^6}$.

Then $z=1$ is pole of order $m=3$ and $z=3$ is a pole of order $m=6$.

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Unit - 4

Functions of complex Variables

Q.1. Using cauchy's integral formula, evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$, where C is the circle $|z|=3$.

Sol. By derivative of Cauchy's integral formula:

$$f^{n}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz, \quad \text{——} (1)$$

$\therefore$ we have $f(z) = e^{2z}$, $a = -1$

here $n+1 = 4 \Rightarrow \boxed{n=3}$

From (1),

$$f^{3}(a) = \frac{3!}{2\pi i} \int_C \frac{e^{2z}}{(z+1)^{3+1}} dz$$

$$\Rightarrow f^{3}(-1) = \frac{3!}{2\pi i} \int_C \frac{e^{2z}}{(z+1)^4} dz \quad \text{——} (2)$$

since $f(z) = e^{2z}$

$\Rightarrow f'(z) = 2e^{2z}$

$\Rightarrow f''(z) = 4e^{2z}$

$\Rightarrow f'''(z) = 8e^{2z}$

$\therefore f^{3}(-1) = 8e^{-2}$

From (2)

$$\int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!} \times 8e^{-2} = \frac{8\pi i}{3e^2}$$

$$\therefore \int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{8\pi i}{3e^2}.$$

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Q.2 Evaluate the integral using Cauchy's integral formula

$$\int_C \frac{e^{2z}}{(z-1)(z-2)} dz, \text{ where } C \text{ is } |z| = 3.$$

Sol. Here $f(z) = e^{2z}$ is analytic within the circle $|z| = 3$ and the two singular points $a = 1$ and 2, lies inside C.

Now,
$$\int_C \frac{e^{2z}}{(z-1)(z-2)} dz = \int_C e^{2z}\left[\frac{1}{z-2} - \frac{1}{z-1}\right] dz. \quad \text{—(1)}$$

because,

$$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$\Rightarrow 1 = A(z-2) + B(z-1) \quad \text{—(2)}$$

put $z = 2$ in (2)

$$1 = B(2-1) \Rightarrow \boxed{B = 1}$$

& put $z = 1$ in (2)

$$1 = A(1-2) \Rightarrow \boxed{A = -1}$$

From (1), we have

$$\int_C \frac{e^{2z}}{(z-1)(z-2)} dz = \int_C \frac{e^{2z}}{z-2} dz - \int_C \frac{e^{2z}}{z-1} dz$$

$$= 2\pi i f(2) - 2\pi i f(1)$$

$$\left[\because \text{By Cauchy's integral formula: } \int \frac{f(z)}{(z-a)} dz = 2\pi i f(a) ; a \in C\right]$$

$$= 2\pi i e^4 - 2\pi i e^2 = 2\pi i e^2 (e^2 - 1).$$

$$\left[\because f(z) = e^{2z} \Rightarrow f(2) = e^4 \text{ \& } f(1) = e^2\right]$$

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Unit - 4

Functions of Complex Variables

Residues of $f(z)$ at pole.

Since analytic function $f(z)$ can be expanded in a Laurent's Series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

Then the coefficient of $\frac{1}{(z-a)}$ i.e, b_1 is called the residues of $f(z)$ at pole $z=a$. It is denoted by

$$\left[\text{Res } f(z)\right]_{z=a} \quad \text{or} \quad \text{Res. } f(a).$$

Methods of Findingout Residues of $f(z)$ at pole.

1. Residue of $f(z)$ at simple pole $z=a$ (i.e pole of order 1):

$$\left[\text{Res } f(z)\right]_{z=a} = \lim_{z \to a} (z-a) f(z)$$

2. Residue of $f(z)$ has pole of order m at $z=a$:

$$\left[\text{Res } f(z)\right]_{z=a,m} = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}.$$

3. Residue of $f(z)$ at infinity:

i.e. Residue of $f(z)$ at $z=\infty$ $= \lim_{z \to \infty} \{-z \cdot f(z)\}$

$$= -\frac{1}{2\pi i} \int_C f(z)\, dz.$$

Cauchy's Residue Theorem.

Statement: If $f(z)$ is analytic function, except at a finite number of poles $a_1, a_2, \ldots, a_n$ within a closed contour C and continuous on the boundary C, then

$$\boxed{\int_C f(z)\,dz = 2\pi i \,(\text{sum of residues at the poles with in } C)}$$

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Q.1. Find the order of each pole and residue at it of

$$\frac{1-2z}{z(z-1)(z-2)}.$$

Sol. Let $f(z) = \frac{1-2z}{z(z-1)(z-2)}$

For the pole of $f(z)$: put $z(z-1)(z-2)=0$

$\Rightarrow z = 0, 1, 2$ all are simple poles.

We know that $\left[\text{Res } f(z)\right]_{z=a} = \lim_{z\to a} (z-a) f(z)$

(i) $\left[\text{Res. } f(z)\right]_{z=0} = \lim_{z\to 0} (z-0) f(z) = \lim_{z\to 0} \cancel{z} \cdot \frac{1-2z}{\cancel{z}(z-1)(z-2)}$

$= \lim_{z\to 0} \frac{1-2z}{(z-1)(z-2)} = \frac{1-2\times 0}{(0-1)(0-2)} = \frac{\cancel{+}1}{\cancel{+}2}$

$= \frac{1}{2}$

(ii) $\left[\text{Res } f(z)\right]_{z=1} = \lim_{z\to 1} (z-1) f(z) = \lim_{z\to 1} \cancel{(z-1)} \cdot \frac{1-2z}{z\cancel{(z-1)}(z-2)}$

$= \lim_{z\to 1} \frac{1-2z}{z(z-2)} = \frac{1-2\times 1}{1(1-2)} = 1$

(iii) $\left[\text{Res } f(z)\right]_{z=2} = \lim_{z\to 2} (z-2) f(z) = \lim_{z\to 2} \cancel{(z-2)} \cdot \frac{1-2z}{z(z-1)\cancel{(z-2)}}$

$= \lim_{z\to 2} \frac{1-2z}{z(z-1)} = \frac{1-2\times 2}{2(2-1)}$

$= -\frac{3}{2}$.

Q.2. Determine the pole of the function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ and the residue at each pole.

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Sol. Let $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$,

For the poles of $f(z)$: put $(z-1)^2(z+2)=0$.

$\Rightarrow z=-2$ is a simple pole and $z=1$ is a pole of order 2.

We know that $\left[\text{Res.} f(z)\right]_{z=a} = \lim_{z\to a}(z-a)f(z)$

(i) $\left[\text{Res.} f(z)\right]_{z=-2} = \lim_{z\to -2}(z+2)f(z) = \lim_{z\to -2}\cancel{(z+2)}\cdot\dfrac{z^2}{(z-1)^2\cancel{(z+2)}}$

$$= \lim_{z\to -2}\frac{z^2}{(z-1)^2} = \frac{(-2)^2}{(-2-1)^2} = \frac{4}{9}.$$

We know that $\left[\text{Res} f(z)\right]_{z=a,m} = \dfrac{1}{(m-1)!}\lim_{z\to a}\dfrac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\}$.

(ii) $\left[\text{Res.} f(z)\right]_{z=1,m=2} = \dfrac{1}{(2-1)!}\lim_{z\to 1}\dfrac{d^{2-1}}{dm^{2-1}}\left\{(z-1)^2 f(z)\right\}$.

$$= \lim_{z\to 1}\frac{d}{dz}\left\{\cancel{(z-1)^2}\cdot\frac{z^2}{\cancel{(z-1)^2}(z+2)}\right\}$$

$$= \lim_{z\to 1}\frac{d}{dz}\left\{\frac{z^2}{(z+2)}\right\}$$

$$= \lim_{z\to 1}\left\{\frac{(z+2).2z - z^2(1)}{(z+2)^2}\right\}$$

$$= \lim_{z\to 1}\frac{(z^2+4z)}{(z+2)^2} = \frac{(1+4)}{(1+2)^2} = \frac{5}{9}.$$

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Q. 3. ~~Find~~ Evaluate $\int_C \frac{(4-3z)}{z(z-1)(z-2)}$, where C is the circle $|z| = 3/2$.

Sol. Let $f(z) = \frac{4-3z}{z(z-1)(z-2)}$.

Using residue theorem:

$$\int_C f(z)\,dz = 2\pi i\,[\Sigma \text{ Res. of } f(z) \text{ at each pole with in } C] \quad (1)$$

For the poles of $f(z)$: put $z(z-1)(z-2) = 0$

$\Rightarrow$ $z = 0, 1, 2$, all are simple poles.

At pole $z = 0$, then $|z| = |0| = 0 < 3/2$, which lie inside C.

At pole $z = 1$, then $|z| = |1| = 1 < 3/2$, which lie inside C.

At pole $z = 2$, then $|z| = |2| = 2 \nless 3/2$, which is not lie inside the circle C.

We know that: $[\text{Res } f(z)]_{z=a} = \lim_{z \to a} (z-a) f(z)$, then

(i) $[\text{Res } f(z)]_{z=0} = \lim_{z \to 0} (z-0) \cdot \frac{(4-3z)}{z(z-1)(z-2)}$

$= \lim_{z \to 0} \frac{(4-3z)}{(z-1)(z-2)} = \frac{4}{2} = 2.$

(ii) $[\text{Res } f(z)]_{z=1} = \lim_{z \to 1} \cancel{(z-1)} \cdot \frac{(4-3z)}{z\cancel{(z-1)}(z-2)} = \frac{(4-3)}{1(1-2)} = -1$

Hence (1) becomes:

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$$\therefore \int_C \frac{(4-3z)}{z(z-1)(z-2)}\,dz = 2\pi i\left[2+(-1)\right] = 2\pi i,$$

Q.4. Evaluate $\int_C \frac{z-3}{z^2+2z+5}\,dz$, where C is the circle $|z+1+i| = 2$.

Sol. Let $f(z) = \frac{z-3}{z^2+2z+5}$

Using residue theorem:

$$\int_C f(z)dz = 2\pi i\left[\Sigma \text{ Res. of } f(z) \text{ at each pole within } C\right] \quad (1)$$

For the poles of $f(z)$:

put $z^2+2z+5=0$

$$\Rightarrow z = \frac{-2\pm\sqrt{4-20}}{2} = \frac{-2\pm 4i}{2}$$

$$\Rightarrow z = -1\pm 2i$$

$\Rightarrow$ $z=-1+2i$ and $z=-1-2i$ are simple poles.

At pole $z=-1+2i$, then $|z+1+i| = |-1+2i+1+i| = |3i|$

$= \sqrt{0+3^2} = 3 \not< 2$, outside in C.

At pole $z=-1-2i$, then $|z+1+i| = |-1-2i+1+i| = |i|$

$= \sqrt{0+1^2} = 1 < 2$, inside in C.

Thus only pole $z=-1-2i$ lie inside in C.

$$\therefore \left[\text{Res } f(z)\right]_{z=-1-2i} = \lim_{z\to -1-2i} (z+1+2i)\, f(z)$$

$$= \lim_{z\to -1-2i} \frac{(z+1+2i)(z-3)}{z^2+2z+5} = \lim_{z\to -1-2i} \frac{(\cancel{z+1+2i})(z-3)}{(\cancel{z+1+2i})(z+1-2i)}$$

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$$= \lim_{z \to -1-2i} \frac{z-3}{z+1-2i} = \frac{-1-2i-3}{-1-2i+1-2i} = \frac{4+2i}{4i}$$

$$= \frac{1}{2} - i$$

Here ① becomes:

$$\int_C \frac{z-3}{z^2+2z+5} dz = 2\pi i \left[\frac{1}{2} - i\right] = \pi(2+i).$$

Q.5 Evaluate $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz$, where C is the circle $|z|=3$.

Sol. Let $f(z) = \frac{\cos \pi z^2}{(z-1)(z-2)}$.

we know that:

$$\int_C f(z) dz = 2\pi i \left[\Sigma \text{ Res. of } f(z) \text{ at each point with in } C\right]. \quad ①$$

For the poles of $f(z)$: put $(z-1)(z-2) = 0$

$\Rightarrow$ $z=1$ and $z=2$ are simple poles.

At pole $z=1$, then $|z| = |1| = 1 < 3$, inside C.

At pole $z=2$, then $|z| = |2| = 2 < 3$, Inside C.

Hence both poles are lie inside of C.

(i) $$[\text{Res } f(z)]_{z=1} = \lim_{z \to 1} \cancel{(z-1)} \cdot \frac{\cos \pi z^2}{\cancel{(z-1)}(z-2)}$$

$$= \lim_{z \to 1} \frac{\cos \pi z^2}{(z-2)} = \frac{\cos \pi}{(-1)} = \frac{-1}{(-1)} = 1.$$

(ii) $$[\text{Res } f(z)]_{z=2} = \lim_{z \to 2} (z-2) \frac{\cos \pi z^2}{(z-1)(z-2)} = \frac{\cos 4\pi}{(2-1)} = \frac{1}{1} = 1.$$

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Hence (1) becomes;

$$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i[1+1] = 4\pi i.$$

Q.6. Evaluate $\int_C \frac{e^z}{\cos \pi z} dz$, where C is the unit circle $|z|=1$.

Sol, Let $f(z) = \frac{e^z}{\cos \pi z}$.

Using residue theorem:

$$\int_C f(z)dz = 2\pi i\left[\Sigma \text{ Res. of } f(z) \text{ at each pole with in } C\right].$$

For the poles of $f(z)$: put $\cos \pi z = 0$

$$\Rightarrow \pi z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

$\Rightarrow z = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$ are all simple poles.

At poles $z = \pm\frac{1}{2}$, then $|z| = |\pm\frac{1}{2}| = \frac{1}{2} < 1$, inside C.

At poles $z = \pm\frac{3}{2}$, then $|z| = |\pm\frac{3}{2}| = \frac{3}{2} \nless 1$, outside C.

Hence only two poles $z=\frac{1}{2}$ and $z=-\frac{1}{2}$ lie inside the circle C,

$$\therefore [\text{Res } f(z)]_{z=\frac{1}{2}} = \lim_{z\to\frac{1}{2}} \left(z-\frac{1}{2}\right)\cdot\frac{e^z}{\cos \pi z},$$

which is $\frac{0}{0}$ from, using L' Hospital rule

$$= \lim_{z\to\frac{1}{2}} \frac{\left[e^z + \left(z-\frac{1}{2}\right)e^z\right]}{-\sin \pi z \cdot \pi} = \frac{e^{\frac{1}{2}}}{\pi\cdot(-\sin \pi/2)} = \frac{-e^{1/2}}{\pi}.$$

$[\because \sin \pi/2 = 1]$

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Similarly,

$$\left[\text{Res } f(z)\right]_{z\to -\frac{1}{2}} = \lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right) \cdot \frac{e^z}{\cos \pi z}, \text{ using L' Hospital rule}$$

$$= \lim_{z\to -\frac{1}{2}} \frac{\left[e^z + \left(z+\frac{1}{2}\right)e^z\right]}{(-\sin \pi z)\pi} = \frac{e^{-1/2}}{(-\pi)\sin(-\pi/2)} = \frac{e^{-1/2}}{(-\pi)(-1)} = \frac{e^{-1/2}}{\pi}$$

Hence (1) becomes:

$$\int_C \frac{e^z}{\cos \pi z} dz = 2\pi i\left[-\frac{e^{1/2}}{\pi} + \frac{e^{-1/2}}{\pi}\right] = -4i\left(e^{1/2} - e^{-1/2}\right)$$

$$= -4i \sinh 1/2$$

Application of Residues to Evaluate Real Integrals.

Integral of the type $\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$, where $F(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$ can be reduced to complex line integrals by the substitution

$$z = e^{i\theta}$$

$$\Rightarrow dz = i e^{i\theta} d\theta \text{ i.e. } d\theta = \frac{dz}{i e^{i\theta}} = \frac{dz}{iz}.$$

Also $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$

and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right)$

As θ varies from 0 to 2π, then z moves one round the unit circle in the anti clockwise direction, i.e. $|z| = 1$.

$$\therefore \int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta = \int_C F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz} = \int_C f(z)dz.$$

where C is the unit circle $|z| = 1$.

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Q.1. Using contour integration evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}.$$

Sol. Let $$I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$$

Put $z = e^{i\theta} \Rightarrow dz = i.e^{i\theta} d\theta \Rightarrow \frac{dz}{e^{i\theta}.i} = d\theta$

$$\Rightarrow d\theta = \frac{dz}{iz}$$

Since $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2} \Rightarrow \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$, then we have

$$I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_C \frac{1}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)} \cdot \frac{dz}{iz}$$

$$= \int_C \frac{2z}{4z+z^2+1} \cdot \frac{dz}{iz}$$

or $$I = \frac{2}{i}\int_C \frac{dz}{(z^2+4z+1)} dz,$$ where C is unit circle $|z|=1$. —— (1)

Let $$f(z) = \frac{1}{z^2+4z+1}$$

For poles: put $z^2+4z+1=0$

$$\Rightarrow z = \frac{-4\pm\sqrt{16-4}}{2} = \frac{-4\pm2\sqrt{3}}{2}$$

$\Rightarrow z = -2\pm\sqrt{3} \Rightarrow z = -2+\sqrt{3}$ and $z = -2-\sqrt{3}$ are simple poles.

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At pole $z = -2+\sqrt{3}$, then $|z| = |-2+\sqrt{3}| = |-2+1.732| = 0.267 < 1$; inside the circle C.

At pole $z = -2-\sqrt{3}$, then $|z| = |-2-\sqrt{3}| = |-2-1.732| = 3.73 \nless 1$, i.e. out side of C.

Thus only the pole $z = -2+\sqrt{3}$ lie inside the unit circle $|z| = 1$.

Hence by Residue theorem;

$$\int_C f(z)\,dz = 2\pi i \left[\text{Res}\, f(-2+\sqrt{3}) \right]$$

$$= 2\pi i \left[\lim_{z \to (-2+\sqrt{3})} \{ z - (-2+\sqrt{3}) \} f(z) \right]$$

$$= 2\pi i \left[\lim_{z \to (-2+\sqrt{3})} \left\{ (z+2-\sqrt{3}) \cdot \frac{1}{z^2+4z+1} \right\} \right]$$

$$= 2\pi i \left[\lim_{z \to (-2+\sqrt{3})} \frac{\cancel{(z+2-\sqrt{3})}}{\cancel{(z+2-\sqrt{3})}(z+2+\sqrt{3})} \right]$$

$$\left[\because z^2+4z+1 = (z+2-\sqrt{3})(z+2+\sqrt{3}) \right]$$

$$= 2\pi i \left[\frac{1}{-2+\sqrt{3}+2+\sqrt{3}} \right] = 2\pi i \left[\frac{1}{2\sqrt{3}} \right] = \frac{\pi i}{\sqrt{3}}.$$

Hence ① becomes;

$$I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i} \int_C f(z)\,dz = \frac{2}{i} \cdot \frac{\pi i}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

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Q.2 Evaluate $\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$.

Sol. Let $I = \int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$

Put $z = e^{i\theta} \Rightarrow dz = i.e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$

Since, $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \Rightarrow \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

$$\therefore I = \int_0^{2\pi} \frac{d\theta}{5+3\sin\theta} = \int_C \frac{1}{5+3\left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]} \cdot \frac{dz}{iz}$$

$$= \frac{1}{i}\int_C \frac{1}{5+3.\frac{1}{2i}\left(\frac{z^2-1}{z}\right)} \cdot \frac{dz}{z}$$

$$= \frac{1}{\cancel{i}}\int_C \frac{2\cancel{i}z}{10iz+3z^2-3} \cdot \frac{dz}{\cancel{z}}$$

$$= 2\int_C \frac{dz}{(3z^2+10iz-3)} = 2\int_C f(z)\,dz \quad \text{--- (1)}$$

where C is the unit circle, i.e. $C : |z| = 1$ and

$$f(z) = \frac{1}{3z^2+10iz-3}$$

For poles: put $3z^2+10iz-3=0$

$$\Rightarrow z = \frac{-10i \pm \sqrt{-100+36}}{6} = \frac{-10i \pm i8}{6}$$

$$\Rightarrow z = \frac{-5i \pm 4i}{3} \quad \text{then}$$

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$\therefore \; z = \frac{-5i+4i}{3} = \frac{-i}{3}$ and $z = \frac{-5i-4i}{3} = -3i$.

At pole; $z = \frac{-i}{3}$, then $|z| = |-i/3| = \sqrt{0+(1/3)^2} = \frac{1}{3} < 1$, inside C.

At pole: $z = -3i$, then $|z| = |-3i| = \sqrt{0+(-3)^2} = 3 \nless 1$, outside C

we know that

$$[\text{Res } f(z)]_{z=-i/3} = \lim_{z \to -\frac{i}{3}} \cancel{\left(z+\frac{i}{3}\right)} \cdot \frac{1}{(z+3i)\cancel{\left(z+\frac{i}{3}\right)}} \cdot \frac{1}{3}$$

$$= \lim_{z \to -i/3} \frac{1}{(z+3i)} \times \frac{1}{3} = \frac{1}{\left(\frac{-i}{3}+3i\right)} \cdot \frac{1}{3}$$

$$= \frac{\cancel{3}}{(-i+9i)} \cdot \frac{1}{\cancel{3}} = \frac{1}{8i}$$

By using residue theorem:

$$\int_C f(z)\,dz = 2\pi i \left(\Sigma \text{ Res } f(z) \text{ at pole } z = -i/3\right)$$

$$= 2\pi i \times \frac{1}{8i} = \frac{2\pi}{8} = \pi/4.$$

Hence ① becomes:

$$I = 2\int_C f(z)\,dz = 2 \cdot \frac{\pi}{4} = \pi/2.$$

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Q.3. Apply the calculus of residues to show that:

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}, \quad a>b>0.$$

Sol Let $$I = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$$

Put $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta \Rightarrow \frac{dz}{ie^{i\theta}} = d\theta$

$$\Rightarrow d\theta = \frac{dz}{iz}$$

Since, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$$\therefore I = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_C \frac{1}{a+b.\frac{1}{2}\left(z+\frac{1}{z}\right)} \cdot \frac{dz}{iz}$$

where C is unit circle $|z|=1$.

$$= \frac{2}{i}\int_C \frac{1}{bz^2+2az+b}\,dz = \frac{2}{ib}\oint_C \frac{1}{z^2+\frac{2a}{b}z+1}\,dz \quad \text{---} (1)$$

For poles of $f(z)$; put $bz^2+2az+b=0$

$$\Rightarrow z = \frac{-2a \pm \sqrt{4a^2-4b^2}}{2b}$$

$$\therefore z = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

Let two pole be;

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$\alpha = \dfrac{-a \pm \sqrt{a^2-b^2}}{b}$ and $\beta = \dfrac{-a-\sqrt{a^2-b^2}}{b}$

since $a > b > 0$, so that $|\beta| > 1$ and $|\alpha| < 1$.

[Because let $a = 2$, $b = 1$, then $\alpha = -2+\sqrt{3} = -0.268$ and $\beta = -3.73$]

$\therefore$ Clearly pole $z = \alpha$ lies inside the unit circle C; but pole $z = \beta$ is out side of C.

$$\therefore \; f(z) = \frac{1}{z^2 + \frac{2az}{b} + 1} = \frac{1}{(z-\alpha)(z-\beta)}$$

$$\therefore \left[\text{Res of } f(z)\right]_{z=\alpha} = \lim_{z\to\alpha} (z-\alpha) f(z)$$

$$= \lim_{z\to\alpha}\left[\cancel{(z-\alpha)} \cdot \frac{1}{\cancel{(z-\alpha)}(z-\beta)}\right]$$

$$= \lim_{z\to\alpha} \frac{1}{z-\beta} = \frac{1}{\alpha-\beta}$$

$$= \frac{1}{\left(\frac{-a\pm\sqrt{a^2-b^2}}{b}\right) - \left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)}$$

$$= \frac{b}{2\sqrt{a^2-b^2}} \quad \text{—— (2)}$$

$\therefore$ By residue theorem;

$$\int_C f(z)\,dz = 2\pi i \cdot \Sigma \,\text{Res}\, f(z) \text{ at pole in C}$$

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$$= 2\pi i \times \frac{b}{2\sqrt{a^2-b^2}} = \frac{\pi i b}{\sqrt{a^2-b^2}}$$

Hence ① becomes:

$$I = \frac{2}{\cancel{ib}} \times \frac{\pi i \cancel{b}}{\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}.$$

Q4. Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta$ by contour integration.

Sol. Let $I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta$

put $z = e^{i\theta} \Rightarrow dz = i e^{i\theta} d\theta$

$$\Rightarrow d\theta = \frac{dz}{i e^{i\theta}} \Rightarrow d\theta = \frac{dz}{iz}.$$

since $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{1}{2}\left(\frac{z^2+1}{z}\right)$

and $\cos 2\theta = \frac{e^{2i\theta}+e^{-2i\theta}}{2} = \frac{1}{2}\left(z^2+\frac{1}{z^2}\right) = \frac{1}{2}\left(\frac{z^4+1}{z^2}\right)$

$$\therefore I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta = \int_C \frac{\frac{1}{2}\left(\frac{z^4+1}{z^2}\right)}{5+2\left(\frac{z^2+1}{z}\right)} \cdot \frac{dz}{iz}$$

where C is the circle $|z| = 1$.

$$= \frac{1}{2i}\int_C \frac{z^4+1}{z^2(2z^2+5z+2)}\, dz = \frac{1}{2i}\int_C f(z)\, dz \quad \text{—①}$$

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where, $f(z) = \dfrac{z^4+1}{z^2(2z^2+5z+2)}$

For the poles of $f(z)$: put $z^2(2z^2+5z+2)=0$

i.e. $z=0$, pole of order 2, and

$$z = \frac{-5 \pm \sqrt{25-16}}{4}$$

i.e. $z = \dfrac{-5+3}{4}, \dfrac{-5-3}{4}$.

$\Rightarrow z = -\frac{1}{2}$ and -2 are simple poles.

At pole $z=0$, then $|z| = |0| = 0 < 1$, inside C.

At pole $z = -\frac{1}{2}$, then $|z| = |-\frac{1}{2}| = \frac{1}{2} < 1$ inside C.

At pole $z=-2$, then $|z| = |-2| = 2 \nless 1$, outside C.

$$\therefore \left[\text{Res } f(z)\right]_{z=0, m=2} = \frac{1}{(2-1)!} \lim_{z \to 0} \frac{d}{dz}\left\{(z-0)^2 f(z)\right\}$$

$$= \lim_{z\to 0} \frac{d}{dz}\left\{z^2 \cdot \frac{(z^4+1)}{z^2(2z^2+5z+2)}\right\}$$

$$= \lim_{z\to 0} \frac{d}{dz}\left\{\frac{z^4+1}{2z^2+5z+2}\right\}$$

$$= \lim_{z\to 0}\left\{\frac{(2z^2+5z+2)(4z^3) - (z^4+1)(4z+5)}{(2z^2+5z+2)^2}\right\}$$

$$= \frac{-5}{4}.$$

and

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$$\left[\operatorname{Res} f(z)\right]_{z=-1/2} = \lim_{z \to -1/2} (z+1/2) \cdot f(z)$$

$$= \lim_{z \to -1/2} \cancel{(z+\tfrac{1}{2})} \cdot \frac{(z^4+1)}{z^2 \cdot 2 \cancel{(z+\frac{1}{2})}(z+2)}$$

$$\left[\because \frac{1}{z^2(2z^2+5z+2)} = \frac{1}{2z^2 (z+\frac{1}{2})(z+2)}\right]$$

$$= \lim_{z \to -\frac{1}{2}} \frac{(z^2+1)}{2z^2 (z+2)} = \frac{17}{12}$$

By using residue theorem:

$$\int_c f(z)\,dz = 2\pi i \times (\text{sum of residue at each pole in c})$$

$$= 2\pi i \left(-\frac{5}{4} + \frac{17}{12}\right) = 2\pi i \times \frac{2}{12} = \frac{\pi i}{3}$$

Hence ① becomes:

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = \frac{1}{2i} \times \frac{\pi i}{3} = \frac{\pi}{6}.$$

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