Equation (2.31) can also be derived by transformation of coordinates as follows :

$$x = r \sin \theta \sin \phi \; ; \; y = r \sin \theta \cos \phi \; ; \; z = r \cos \theta$$

2.5. HEAT CONDUCTION THROUGH PLANE AND COMPOSITE WALLS

2.5.1. HEAT CONDUCTION THROUGH A PLANE WALL

Case I : Uniform thermal conductivity

Refer to Fig. 2.4 (*a*) Consider a plane wall of homogeneous material through which heat is flowing *only in x-direction*.

Let, L = Thickness of the plane wall,

A = Cross-sectional area of the wall,

k = Thermal conductivity of the wall material, and

t_1, t_2 = Temperatures maintained at the two faces 1 and 2 of the wall, respectively.

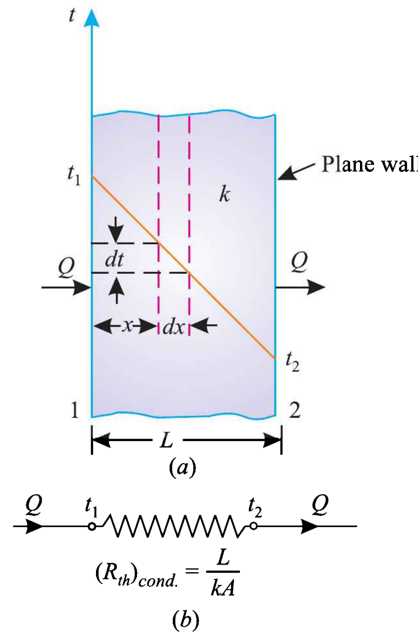


Fig. 2.4. Heat conduction through a plane wall.

The general heat conduction equation in cartesian coordinates is given by

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} + \frac{q_g}{k} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \quad \text{...[Eqn. 2.8]}$$

If the heat conduction takes place under the conditions, steady state $\left(\frac{\partial t}{\partial \tau} = 0\right)$, one-dimensional $\left[\frac{\partial^2 t}{\partial y^2} = \frac{\partial^2 t}{\partial z^2} = 0\right]$ and with no internal heat generation $\left(\frac{q_g}{k} = 0\right)$ then the above equation is reduced to

$$\frac{\partial^2 t}{\partial x^2} = 0, \quad \text{or} \quad \frac{d^2 t}{dx^2} = 0 \quad \text{...(2.33)}$$

By integrating the above differential twice, we have

$$\frac{dt}{dx} = C_1 \quad \text{and} \quad t = C_1 x + C_2 \quad \text{...(2.34)}$$

where C_1 and C_2 are the arbitrary constants. The values of these constants may be calculated from the known boundary conditions as follows :

At $x = 0$ $\quad t = t_1$

At $x = L$ $\quad t = t_2$

Substituting the values in the eqn. (2.34), we get

$$t_1 = O + C_2 \quad \text{and} \quad t_2 = C_1 L + C_2$$

After simplification, we have, $C_2 = t_1$ and $C_1 = \frac{t_2 - t_1}{L}$

Thus, the eqn. (2.34) reduces to :

$$t = \left(\frac{t_2 - t_1}{L}\right) x + t_1 \quad \text{...(2.35)}$$

The eqn. (2.35) indicates that *temperature distribution across a wall is linear* and is *independent of thermal conductivity*. Now heat through the plane wall can be found by using Fourier's equation as follows :

$$Q = -kA\frac{dt}{dx} \qquad \left(\text{where, } \frac{dt}{dx} = \text{Temperature gradient}\right) \quad \ldots \text{[Eqn.(1.1)]}$$

But,
$$\frac{dt}{dx} = \frac{d}{dx}\left[\left(\frac{t_2 - t_1}{L}\right)x + t_1\right] = \frac{t_2 - t_1}{L}$$

∴
$$Q = -kA\frac{(t_2 - t_1)}{L} = \frac{kA\,(t_1 - t_2)}{L} \quad \ldots(2.36)$$

Eqn (2.36) can be written as :

$$Q = \frac{(t_1 - t_2)}{(L/kA)} = \frac{(t_1 - t_2)}{(R_{th})_{cond.}} \quad \ldots(2.37)$$

where, $(R_{th})_{cond.}$ = Thermal resistance to heat conduction. Fig. 2.4 (*b*) shows the *equivalent thermal circuit* for heat flow through the plane wall.

Let us now find out the condition when instead of space, weight is the main criterion for selection of the insulation of a plane wall.

Thermal resistance (conduction) of the wall, $(R_{th})_{cond.} = \frac{L}{kA}$ ...(*i*)

Weight of the wall, $W = \rho A L$ ...(*ii*)

Eliminating L from (*i*) and (*ii*), we get

$$W = \rho\, A.\, (R_{th})_{cond.}\, kA = (\rho.k)A^2.(R_{th})_{cond.} \quad \ldots(2.38)$$

The eqn., (2.38) stipulates the condition that, for a specified thermal resistance, the *lightest insulation will be one which has the smallest product of density* (ρ) *and thermal conductivity (k).*

Case II. Variable thermal conductivity

A. *Temperature variation in terms of surface temperatures* (t_1, t_2) :

A diesel engine is more efficient due to internal combustion and better heat.

Let the thermal conductivity vary with temperature according to the relation

$$k = k_0 (1 + \beta t) \qquad ...(2.39)$$

[In most of the cases, the thermal conductivity is found to vary *linearly with temperature*]

where, k_0 = Thermal conductivity at zero temperature.

When the effect of temperature on thermal conductivity is considered, the Fourier's equation,

$$Q = -kA \frac{dt}{dx} \text{ is written as :}$$

$$Q = -k_0 (1 + \beta t) \frac{dt}{dx} . A \qquad ...(2.40)$$

or, $$\frac{Q}{A} \cdot dx = -k_0 (1 + \beta t)\, dt$$

or, $$\frac{Q}{A} \int_0^L dx = -k_0 \int_{t_1}^{t_2} (1 + \beta t)\, dt$$

or, $$\frac{Q.L}{A} = -k_0 \left[t + \frac{\beta}{2} t^2 \right]_{t_1}^{t_2}$$

or, $$\frac{Q.L}{A} = -k_0 \left[(t_2 - t_1) + \frac{\beta}{2} (t_2^2 - t_1^2) \right] \qquad ...(2.41)$$

$$= k_0 \left[(t_1 - t_2) + \frac{\beta}{2} (t_1 - t_2)(t_1 + t_2) \right]$$

$$= k_0 \left[1 + \frac{\beta}{2} (t_1 + t_2) \right] (t_1 - t_2)$$

$$= k_0 [1 + \beta t_m] (t_1 - t_2) \qquad \text{where } t_m = \frac{t_1 + t_2}{2}$$

∴ $$Q = k_0 (1 + \beta t_m) . \frac{A (t_1 - t_2)}{L}$$

From eqn. (2.39) t is replaced by t_m, then

$$k_m = k_0 (1 + \beta t_m) \qquad ...(2.42)$$

∴ $$Q = k_m A \left[\frac{t_1 - t_2}{L} \right] \qquad ...(2.43)$$

where k_m is known as *mean thermal conductivity* of the wall material.

Further, if t is the temperature of the surface at a distance x from the left surface (Fig. 2.5), then eqn. (2.41) becomes

$$\frac{Qx}{A} = -k_0 \left[(t - t_1) + \frac{\beta}{2} (t^2 - t_1^2) \right] \qquad ...(2.44)$$

Form eqns. (2.41) and (2.44), we have

$$\left[(t_2 - t_1) + \frac{\beta}{2} (t_2^2 - t_1^2) \right] \frac{X}{L} = \left[(t - t_1) + \frac{\beta}{2} (t^2 - t_1^2) \right]$$

[Equating the values of Q and rearranging]

Solving the above equation for t, we get

$$t = \frac{1}{\beta} \left[(1 + \beta t_1)^2 - \{(1 + \beta t_1)^2 - (1 + \beta t_2)^2\} \frac{x}{L} \right]^{1/2} - \frac{1}{\beta} \qquad ...(2.45)$$

B. *Temperature variation in terms of heat flux* (Q) :

Fourier's equation for heat conduction is given by

$$Q = -kA.\frac{dt}{dx} = -k_0\ (1 + \beta t)\ A.\frac{dt}{dx}$$

or, $$Q.dx = -k_0\,(1 + \beta t)\,A.dt$$

Integrating both sides, we get

$$Q.x = -k_0 A\left(t + \frac{\beta}{2}t^2\right) + C \quad ...(i)$$

(where, C = Constant of integration)

To evaluate C, applying the condition : At $x = 0$, $t = t_1$, we get

$$C = k_0 A\left(t_1 + \frac{\beta}{2}t_1^2\right)$$

Substituting the values of the constant C in (i), we get

$$Q.x = -k_0 A\left(t + \frac{\beta}{2}t^2\right) + k_0 A\left(t_1 + \frac{\beta}{2}t_1^2\right)$$

Dividing both sides by k_0A and rearranging, we obtain,

$$\frac{\beta}{2}t^2 + t + \left[\frac{Q.x}{k_0 A} - \left(t_1 + \frac{\beta}{2}t_1^2\right)\right] = 0$$

By solving the above quadratic equation, we have

$$\therefore \quad t = \frac{-1 + \sqrt{1 - 4 \times \frac{\beta}{2}\left[\frac{Q.x}{k_0 A} - \left(t_1 + \frac{\beta}{2}t_1^2\right)\right]}}{2 \times \left(\frac{\beta}{2}\right)}$$

or, $$t = -\frac{1}{\beta} + \left[\frac{1}{\beta^2} - \frac{2}{\beta}\left(\frac{Q.x}{k_0 A} - t_1 - \frac{\beta}{2}t_1^2\right)\right]^{1/2}$$

$$= -\frac{1}{\beta} + \left[\frac{1}{\beta^2} - \frac{2}{\beta}t_1 + t_1^2 - \frac{2Q.x}{\beta k_0 A}\right]^{1/2}$$

$$= -\frac{1}{\beta} + \left[\left(t_1 + \frac{1}{\beta}\right)^2 - \frac{2Q.x}{\beta k_0 A}\right]^{1/2}$$

Hence,

$$t = -\frac{1}{\beta} + \left[\left(t_1 + \frac{1}{\beta}\right)^2 - \frac{2Q.x}{\beta k_0 A}\right]^{1/2} \quad ...(2.46)$$

In most of the practical applications where the variation of temperature is small, the average value of k for the given temperature range is commonly used as given in eqn. (2.42).

If the variation of k with temperature is *not linear*, then

$$k = k_0 f(t), \text{ and}$$

$$\frac{Q}{A}\int_0^L dx = -\int_{t_1}^{t_2} [k_0\ f\ (t)\ dt$$

or, $$Q = \frac{A}{L}\left[-\int_{t_1}^{t_2} [k_0\ f\ (t)\ dt\right] \quad ...(2.47)$$

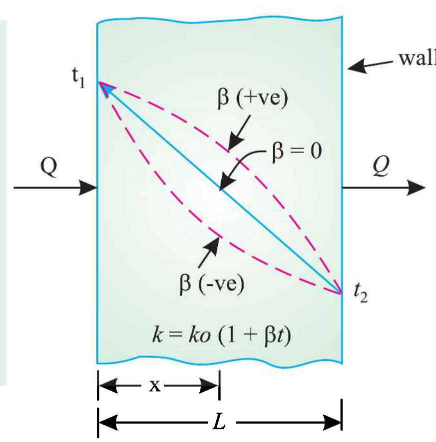


Fig. 2.5.

But, $$Q = k_m A \left(\frac{t_1 - t_2}{L} \right) \qquad \text{...[Eqn. (2.43)]}$$

Equating these eqns. (2.47) and (2.43), we have

$$k_m = \frac{1}{(t_1 - t_2)} \left[\int_{t_1}^{t_2} [k_0 f(t)\, dt \right]$$

$$= \frac{1}{(t_1 - t_2)} \int_{t_2}^{t_1} [k_0 f(t)\, dt] \qquad \text{...(2.48)}$$

The effect of + β and – β on temperature is depicted in Fig. 2.5.

2.5.2. HEAT CONDUCTION THROUGH A COMPOSITE WALL

Refer to Fig. 2.6 (*a*). Consider the transmission of heat through a composite wall consisting of a number of slabs.

Let, L_A, L_B, L_C = Thicknesses of slabs *A*, *B* and *C* respectively (also called path lengths),

k_A, k_B, k_C = Thermal conductivities of the slabs *A*, *B*, and *C* respectively,

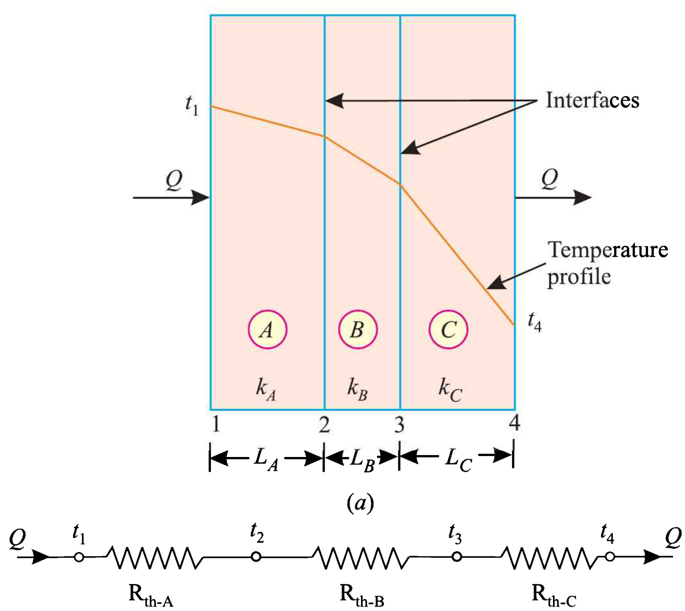
$t_1, t_4\ (t_1 > t_4)$ = Temperatures at the wall surfaces 1 and 4 respectively, and

t_2, t_3 = Temperatures at the interfaces 2 and 3 respectively.

Since the quantity of heat transmitted per unit time through each slab/layer is same, we have,

$$Q = \frac{k_A . A\, (t_1 - t_2)}{L_A} = \frac{k_B\, .\, A\, (t_2 - t_3)}{L_B} = \frac{k_C\, .\, A\, (t_3 - t_4)}{L_C}$$

(Assuming that there is a perfect contact between the layers and no temperature drop occurs across the interface between the materials).



$$R_{th.A} = \frac{L_A}{k_A . A}, \quad R_{th.B} = \frac{L_B}{k_B . A}, \quad R_{th.C} = \frac{L_C}{k_C . A}$$

(*b*)

Fig. 2.6. Steady state conduction through a composite wall.

Rearranging the above expression, we get

$$t_1 - t_2 = \frac{Q \cdot L_A}{k_A \cdot A} \quad \text{...}(i)$$

$$t_2 - t_3 = \frac{Q \cdot L_B}{k_B \cdot A} \quad \text{...}(ii)$$

$$t_3 - t_4 = \frac{Q \cdot L_c}{k_C \cdot A} \quad \text{...}(iii)$$

Adding (*i*), (*ii*) and (*iii*), we have

$$(t_1 - t_4) = Q\left[\frac{L_A}{k_A \cdot A} + \frac{L_B}{k_B \cdot A} + \frac{L_C}{k_C \cdot A}\right]$$

or,

$$Q = \frac{A\,(t_1 - t_4)}{\left[\frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C}\right]} \quad \text{...(2.49)}$$

or,

$$Q = \frac{(t_1 - t_4)}{\left[\frac{L_A}{k_A \cdot A} + \frac{L_B}{k_B \cdot A} + \frac{L_C}{k_C \cdot A}\right]} = \frac{(t_1 - t_4)}{[R_{th-A} + R_{th-B} + R_{th-C}]} \quad \text{...[2.49 (a)]}$$

If the composite wall consists of *n* slabs/layers, then

$$Q = \frac{[t_1 - t_{(n+1)}]}{\sum_1^n \frac{L}{kA}} \quad \text{...(2.50)}$$

In order to solve more complex problems involving both series and parallel thermal resistances, the electrical analogy may be used. A typical problem and its analogous electric circuit are shown in Fig. 2.7.

$$Q = \frac{\Delta t_{\text{overall}}}{\sum R_{th}} \quad \text{...(2.51)}$$

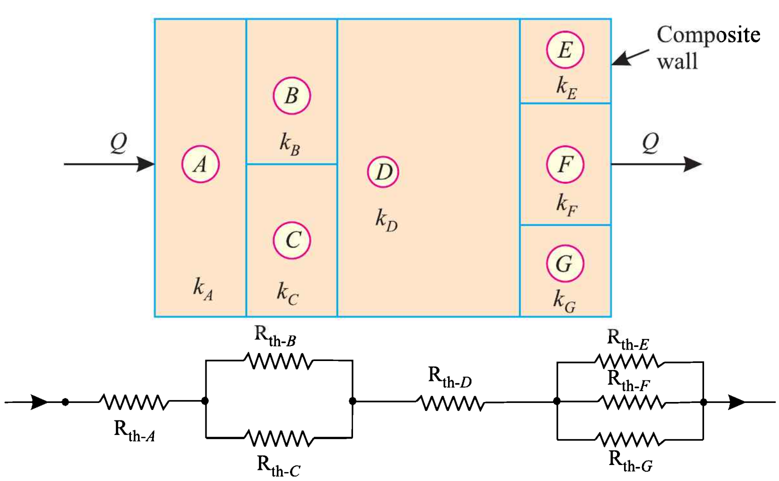


Fig. 2.7. Series and parallel one-dimensional heat transfer through a composite wall and electrical analog.

Thermal contact resistance. In a composite (multi-layer) wall, the calculations of heat flow are made on the assumptions : (*i*) The contact between the adjacent layers is perfect, (*ii*) At the interface there is no fall of temperature, and (*iii*) At the interface the temperature is continuous, although there is discontinuity in temperature gradient. In real systems, however, due to surface roughness and void spaces (usually filled with air) the contact surfaces *touch only at discrete locations*. Thus there is not a single plane of contact, which means that the area available for the flow of heat at the interface will be small compared to geometric face area. Due to this reduced area and presence of air voids, *a large resistance to heat flow at the interface occurs*. This resistance is known as *thermal contact resistance* and it causes temperature drop between two materials at the interface as shown in Fig. 2.8.

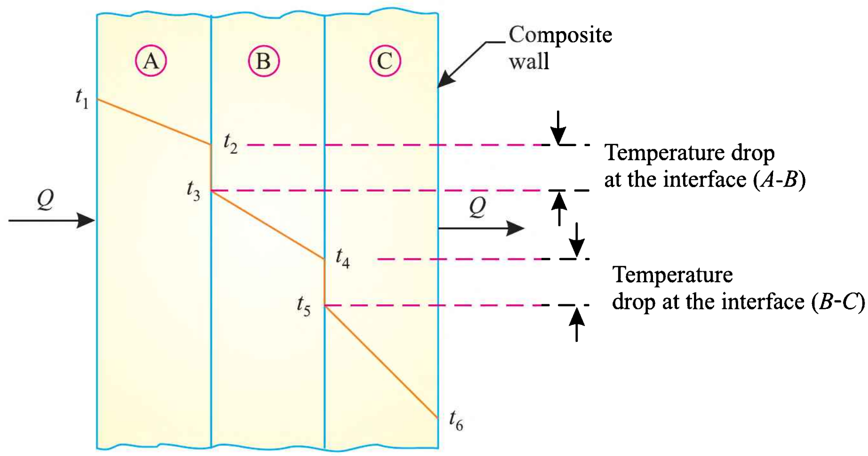


Fig. 2.8. Temperature drops at the interfaces.

Refer to Fig. 2.8. The contact resistances are given by

$$(R_{th-AB})_{cont.} = \frac{(t_2 - t_3)}{Q/A} \quad \text{and} \quad (R_{th-BC})_{cont.} = \frac{(t_4 - t_5)}{Q/A}$$

Boiler is being transported.

2.5.3. THE OVERALL HEAT-TRANSFER COEFFICIENT

While dealing with the problems of fluid to fluid heat transfer across a metal boundary, it is usual to adopt an overall heat transfer coefficient U which *gives the heat transmitted per unit area per unit time per degree temperature difference between the bulk fluids on each side of the metal.*

Refer to Fig. 2.9

Let, L = Thickness of the metal wall,

k = Thermal conductivity of the wall material,

t_1 = Temperature of the surface–1,

t_2 = Temperature of the surface–2,

t_{hf} = Temperature of the hot fluid,

t_{cf} = Temperature of the cold fluid,

h_{hf} = Heat transfer coefficient from hot fluid to metal surface, and

h_{cf} = Heat transfer coefficient from metal surface to cold fluid.

(The suffices *hf* and *cf* stand for hot fluid and cold fluid respectively.)

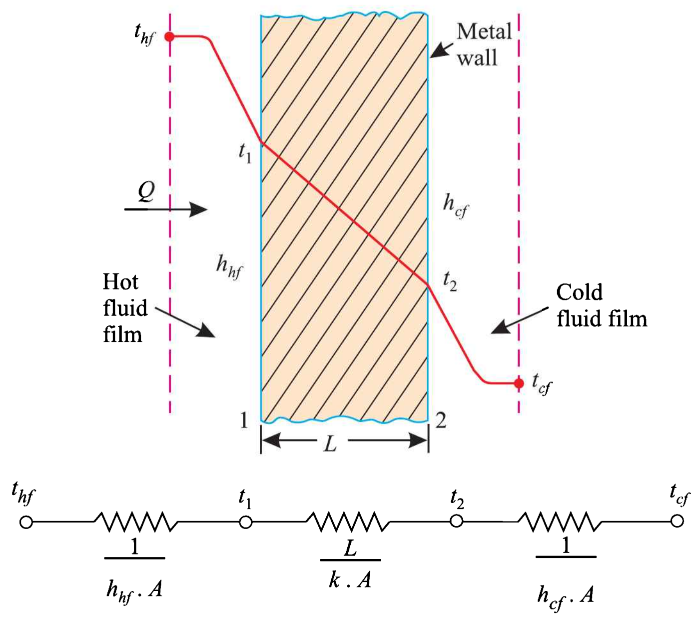


Fig. 2.9. The overall heat transfer through a plane wall.

The equations of heat flow through the fluid and the metal surface are given by

$$Q = h_{hf} A (t_{hf} - t_1) \quad \text{...}(i)$$

$$Q = \frac{k.A (t_1 - t_2)}{L} \quad \text{...}(ii)$$

$$Q = h_{cf} A (t_2 - t_{cf}) \quad \text{...}(iii)$$

By rearranging (*i*), (*ii*) and (*iii*), we get

$$t_{hf} - t_1 = \frac{Q}{h_{hf} \, .A} \quad \text{...}(iv)$$

$$t_1 - t_2 = \frac{QL}{k \, .A} \quad \text{...}(v)$$

$$t_2 - t_{cf} = \frac{Q}{k_{cf} \cdot A} \qquad ...(vi)$$

Adding (*iv*), (*v*) and (*vi*) we get

$$t_{hf} - t_{cf} = Q\left[\frac{1}{h_{hf} \cdot A} + \frac{L}{k \cdot A} + \frac{1}{h_{cf} \cdot A}\right]$$

or,
$$Q = \frac{A\,(t_{hf} - t_{cf})}{\frac{1}{h_{hf}} + \frac{L}{k} + \frac{1}{h_{cf}}} \qquad ...(2.52)$$

If U is the overall coefficient of heat transfer, then

$$Q = U.A\,(t_{hf} - t_{cf}) = \frac{A\,(t_{hf} - t_{cf})}{\frac{1}{h_{hf}} + \frac{L}{k} + \frac{1}{h_{cf}}}$$

or,
$$U = \frac{1}{\frac{1}{h_{hf}} + \frac{L}{k} + \frac{1}{h_{cf}}} \qquad ...(2.53)$$

It may be noticed from the above equation that if the individual coefficients differ greatly in magnitude only a change in the *least* will have any significant effect on the rate of heat transfer.

Example 2.1. *Discuss the effects of various parameters on the thermal conductivity of solids.*

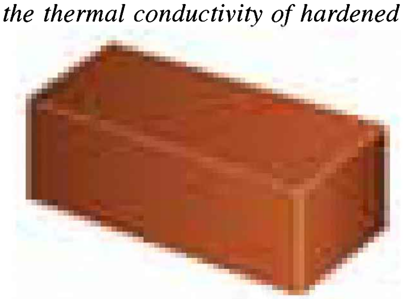
(AMIE Summer, 2001)

Solution. The following are the effects of various parameters on the thermal conductivity of solids.

1. Chemical composition. *Pure metals have very high thermal conductivity. Impurities or alloying elements reduce the thermal conductivity considerably.* [Thermal conductivity of pure copper is 385 W/m° C, and that for pure nickel is 93 W/m° C. But monel metal (an alloy of 30% Ni and 70% Cu) has k of 24 W/m° C. Again for copper containing traces of Arsenic the value of k is reduced to 142 W/m° C.]

2. Mechanical forming. Forging, drawing and bending or *heat treatment of metals* cause considerable variation in thermal conductivity. For example, *the thermal conductivity of hardened steel is lower than that of annealed state.*

3. Temperature rise. The value of k for most metals *decreases with temperature rise* since at elevated temperatures the thermal vibrations of the lattice become higher that retard the motion of free electrons.



Fire brick

4. Non-metallic solids. Non-metallic solids have k *much lower* than that for metals. For many of the building materials (concrete, stone, brick, glass wool, cork etc.) the thermal conductivity may vary from sample to sample due to variations in structure, composition, density and porosity.

5. Presence of air. The thermal conductivity is *reduced* due to the presence of air filled pores or cavities.

6. Dampness. Thermal conductivity of a damp material is *considerably higher* than that of dry material.

7. Density. Thermal conductivity of insulating powder, asbestos etc. increases with density growth. Thermal conductivity of snow is also proportional to its density.