or, $$\frac{\partial^2 t}{\partial x^2} = \cos\phi.\frac{\partial}{\partial r}\left(\frac{\partial t}{\partial x}\right) - \frac{\sin\phi}{r}\cdot\frac{\partial}{\partial\phi}\left(\frac{\partial t}{\partial x}\right)$$

$$= \cos\phi.\frac{\partial}{\partial r}\left(\cos\phi.\frac{\partial t}{\partial r} - \frac{\sin\phi}{r}\cdot\frac{\partial t}{\partial\phi}\right) - \frac{\sin\phi}{r}\cdot\frac{\partial}{\partial\phi}\left(\cos\phi\cdot\frac{\partial t}{\partial r} - \frac{\sin\phi}{r}\cdot\frac{\partial t}{\partial\phi}\right)$$

[Substituting the value of $\frac{\partial t}{\partial x}$ from (*iii*)]

$$= \cos^2\phi.\frac{\partial^2 t}{\partial r^2} - \frac{\cos\phi\,.\sin\phi}{r^2}\cdot\frac{\partial t}{\partial\phi} + \frac{\sin^2\phi}{r}\cdot\frac{\partial t}{\partial r} + \frac{\sin^2\phi}{r^2}\cdot\frac{\partial^2 t}{d\phi^2} + \frac{\sin\phi.\cos\phi}{r^2}\cdot\frac{\partial t}{\partial\phi} \quad ...(iv)$$

Similarly, $$\frac{\partial^2 t}{\partial y^2} = \sin^2\phi.\frac{\partial^2 t}{\partial r^2} + \frac{\cos^2\phi}{r}\cdot\frac{\partial t}{\partial r} - \frac{\cos\phi\,.\sin\phi}{r^2}\cdot\frac{\partial t}{\partial\phi} + \frac{\cos^2\phi}{r^2}\cdot\frac{\partial^2 t}{\partial\phi^2} - \frac{\cos\phi.\sin\phi}{r^2}\cdot\frac{\partial t}{\partial\phi} \quad ...(v)$$

By adding (*iii*) and (*iv*), we get

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = \frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial t}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 t}{\partial\phi^2}$$

Substituting it in eqn (2.8), we get,

$$\left[\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial t}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}\right] + \frac{q_g}{k} = \frac{1}{\alpha}\cdot\frac{\partial t}{\partial\tau}$$

which is the same as eqn. (2.22)

2.4. GENERAL HEAT CONDUCTION EQUATION IN SPHERICAL COORDINATES

Consider an elemental volume having the coordinates (r, ϕ, θ), for three dimensional heat conduction analysis, as shown in Fig. 2.3.

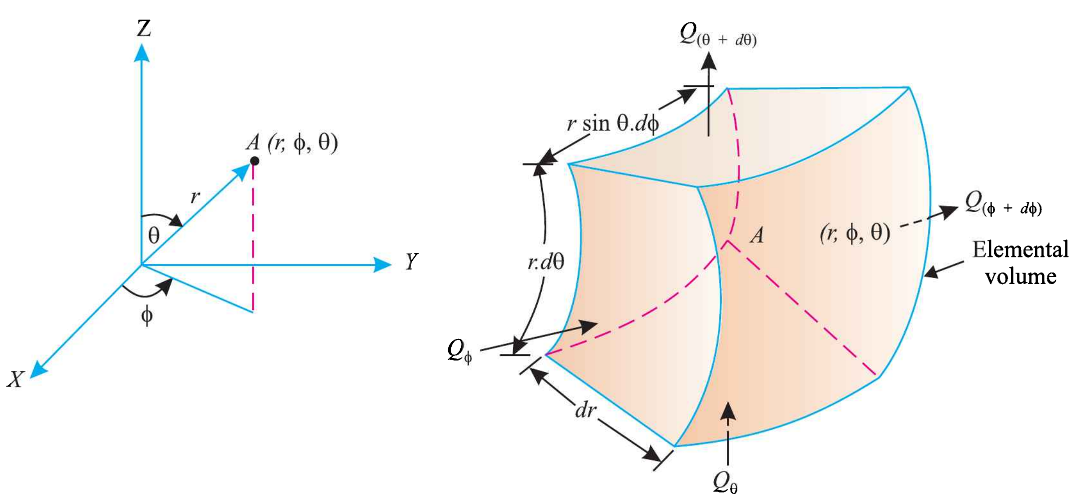

Fig. 2.3. Elemental volume for three-dimensional heat conduction analysis - Spherical coordinates.

The volume of the element $= dr.rd\theta.r\sin\theta\, d\phi$

Let, q_g = Heat generation (uniform) per unit volume per unit time.

Further let us assume that k (thermal conductivity), ρ (density), c (specific heat) do not alter with position.

A. *Net heat accumulated in the element due to conduction of heat from all the coordinate directions considered* :

Heat flow through r–θ *plane*; ϕ-*direction* :

Heat influx, $$Q'_\phi = -k\,(dr.rd\theta)\,\frac{\partial t}{r.\sin\theta.\partial\phi}\,d\tau \qquad ...(i)$$

Heat efflux, $$Q'_{(\phi + d\phi)} = Q'_\phi + \frac{\partial}{r.\sin\theta.\partial\phi}\,(Q'_\phi)\;r\,\sin\theta.d\phi \qquad ...(ii)$$

∴ Heat accumulated in the element due to heat flow in the ϕ-direction,

$$dQ'_\phi = Q'_\phi - Q'_{(\phi + d\phi)} \qquad \text{[subtracting } (ii) \text{ from } (i)]$$

$$= -\frac{1}{r\,\sin\theta}\cdot\frac{\partial}{\partial\phi}\,(Q'_\phi)\;r\,\sin\theta.d\phi$$

$$= -\frac{1}{r\,\sin\theta}\cdot\frac{\partial}{\partial\phi}\left[-k\,(dr.rd\theta)\,\frac{1}{r\,\sin\theta}\cdot\frac{\partial t}{\partial\phi}.\,d\tau\right]r\,\sin\theta.d\phi$$

$$= k\,(dr.rd\theta.r\,\sin\theta.d\phi)\,\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2 t}{\partial\phi^2}\,d\tau \qquad ...(2.25)$$

Heat flow in r–ϕ *plane*, θ-*direction* :

Heat influx, $$Q'_\theta = -\,k\,(dr.\,r\,\sin\,\theta.\,d\phi)\,\frac{\partial t}{r\partial\theta}\cdot d\tau \qquad ...(iii)$$

Heat efflux, $$Q'_{(\theta + d\theta)} = Q'_\theta + \frac{\partial}{r\partial\theta}\,(Q'_\theta)\;rd\theta \qquad ...(iv)$$

∴ Heat accumulated in the element due to heat flow in the θ-direction,

$$dQ'_\theta = Q'_\theta - Q'_{(\theta + d\theta)} \qquad \text{[subtracting } (iv) \text{ from } (iii)]$$

$$= -\frac{\partial}{r.\partial\theta}\,(Q'_\theta)\;r.d\theta$$

$$= -\frac{\partial}{r.\partial\theta}\left[-\,k\,(dr.r\,\sin\theta.d\phi)\,\frac{\partial t}{r.\partial\theta}.\,d\tau\right]r.d\theta$$

Spherical vessels.

$$= \frac{k}{r} \frac{dr.rd\phi.rd\theta}{r} \frac{\partial}{\partial\theta}\left[\sin\theta \cdot \frac{\partial t}{\partial\theta}\right] d\tau$$

$$= k\,(dr.rd\theta.r\sin\theta.d\phi)\,\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left[\sin\theta\cdot\frac{\partial t}{\partial\theta}\right] d\tau \qquad ...(2.26)$$

Heat flow in θ-ϕ *plane*, *r-direction* :

Heat influx, $$Q'_r = -\,k\,(rd\theta.r\sin\theta.d\phi)\,\frac{\partial t}{\partial r}\cdot\partial\tau \qquad ...(v)$$

Heat efflux, $$Q'_{(r+dr)} = Q'_r + \frac{\partial}{\partial r}(Q'_r)\,dr \qquad ...(vi)$$

∴ Heat accumulation in the element due to heat flow in the *r-direction*,

$$dQ'_r = Q'_r - Q'_{(r+dr)} \qquad \text{[subtracting } (vi) \text{ from } (v)]$$

$$= -\frac{\partial}{\partial r}(Q'_r)\,dr$$

$$= -\frac{\partial}{\partial r}\left[-\,k\,(rd\theta.r\sin\theta.d\phi)\,\frac{\partial t}{\partial r}\cdot d\tau\right] dr$$

$$= k\,d\theta.\sin\theta.d\phi\;dr\,\frac{\partial}{\partial r}\left[r^2\cdot\frac{\partial t}{\partial r}\right] d\tau$$

$$= k\,(dr.\,rd\theta.r\sin\theta.d\phi)\,\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left[r^2\cdot\frac{\partial t}{\partial r}\right] d\tau \qquad ...(2.27)$$

Net heat accumulated in the element

$$= k\;dr.rd\theta.r\sin\theta.d\phi\left[\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2 t}{\partial\phi^2} + \frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta.\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2.\frac{\partial t}{\partial r}\right)\right] d\tau \qquad ...(2.28)$$

B. *Heat generated within the element* (Q'_g) :

The total heat generated within the element is given by,

$$Q'_g = q_g\,(dr.rd\theta.r\sin\theta.d\phi)\,d\tau \qquad ...(2.29)$$

C. *Energy stored in the element* :

The increase in thermal energy in the element is equal to

$$\rho\,(dr.rd\theta\,.\,r\sin\theta.d\phi)\,c.\frac{\partial t}{\partial\tau}.\,d\tau \qquad ...(2.30)$$

Now, $(A) + (B) = (C)$ ...Energy balance/equation

$$\therefore\quad k\,dr.rd\theta.r\sin\theta.d\phi\left[\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2 t}{\partial\phi^2} + \frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta.\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2.\frac{\partial t}{\partial r}\right)\right].\,d\tau$$

$$+\;q_g\,(dr.rd\theta.r\sin\theta.d\phi)\,d\tau = \rho\,(dr.rd\theta.r\sin\theta.d\phi)\,c.\frac{\partial t}{\partial\tau}.d\tau$$

Dividing both sides by $k.(dr.rd\theta.\,r\sin\theta.d\phi)d\tau$, we get

$$\left[\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2 t}{\partial\phi^2} + \frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\,.\,\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\,.\,\frac{\partial t}{\partial r}\right)\right] + \frac{q_g}{k}$$

$$= \frac{\rho c}{k}\cdot\frac{\partial t}{\partial\tau} = \frac{1}{\alpha}\cdot\frac{\partial t}{\partial\tau} \qquad ...(2.31)$$

Equation (2.31) is the **general heat conduction equation in spherical coordinates**.

In case there are not *heat sources present* and the heat flow is *steady* and *one-dimensional*, then eqn. (2.31) reduces to

$$\frac{1}{r^2}\cdot\frac{d}{dr}\left(r^2\cdot\frac{dt}{dr}\right) = 0 \qquad ...(2.32)$$

Equation (2.31) can also be derived by transformation of coordinates as follows :

$$x = r \sin \theta \sin \phi \; ; \; y = r \sin \theta \cos \phi \; ; \; z = r \cos \theta$$

2.5. HEAT CONDUCTION THROUGH PLANE AND COMPOSITE WALLS

2.5.1. HEAT CONDUCTION THROUGH A PLANE WALL

Case I : Uniform thermal conductivity

Refer to Fig. 2.4 (*a*) Consider a plane wall of homogeneous material through which heat is flowing *only in x-direction*.

Let,
L = Thickness of the plane wall,
A = Cross-sectional area of the wall,
k = Thermal conductivity of the wall material, and
t_1, t_2 = Temperatures maintained at the two faces 1 and 2 of the wall, respectively.

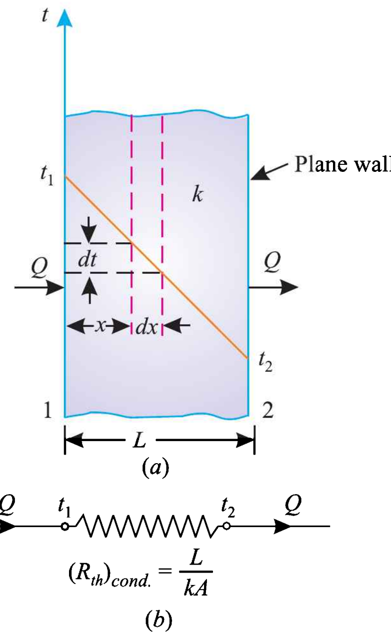

Fig. 2.4. Heat conduction through a plane wall.

The general heat conduction equation in cartesian coordinates is given by

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} + \frac{q_g}{k} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \quad \text{...[Eqn. 2.8]}$$

If the heat conduction takes place under the conditions, steady state $\left(\frac{\partial t}{\partial \tau} = 0\right)$, one-dimensional $\left[\frac{\partial^2 t}{\partial y^2} = \frac{\partial^2 t}{\partial z^2} = 0\right]$ and with no internal heat generation $\left(\frac{q_g}{k} = 0\right)$ then the above equation is reduced to

$$\frac{\partial^2 t}{\partial x^2} = 0, \quad \text{or} \quad \frac{d^2 t}{dx^2} = 0 \quad \text{...(2.33)}$$

By integrating the above differential twice, we have

$$\frac{dt}{dx} = C_1 \quad \text{and} \quad t = C_1 x + C_2 \quad \text{...(2.34)}$$

where C_1 and C_2 are the arbitrary constants. The values of these constants may be calculated from the known boundary conditions as follows :

At $x = 0$ $\quad t = t_1$

At $x = L$ $\quad t = t_2$

Substituting the values in the eqn. (2.34), we get

$$t_1 = O + C_2 \quad \text{and} \quad t_2 = C_1 L + C_2$$

After simplification, we have, $C_2 = t_1$ and $C_1 = \frac{t_2 - t_1}{L}$

Thus, the eqn. (2.34) reduces to :

$$t = \left(\frac{t_2 - t_1}{L}\right) x + t_1 \quad \text{...(2.35)}$$