CHAPTER 3

Stress and Strain

The concepts of stress and strain are essential to design as they characterize the mechanical properties of deformable solids. A brief introduction to the concepts along with a discussion of theories of failure are provided in this chapter. Stress–strain formulas are given for bars subjected to extension, torsion, and bending.

3.1 NOTATION

The units for most of the definitions are given in parentheses, using L for length and F for force.

A Cross-sectional area (L^2)

A_0 Original area; shear-related area defined in Fig. 3-29 (L^2)

A^* Area enclosed by middle line of wall of closed thin-walled cross section (L^2)

b Width (L)

c	Distance from centroidal (neutral) axis of beam to outermost fiber (L)
E	Modulus of elasticity, Young's modulus (F/L^2)
F	Internal force (F)
G	Shear modulus (F/L^2)
I	Moment of inertia of a member about its centroidal (neutral) axis (L^4)
J	Torsional constant; polar moment of inertia for circular cross section (L^4)
L	Length of element, original length (L)
L_s	Total length of middle line of wall of tube cross section (L)
M	Bending moment (FL)
p	Pressure (F/L^2)
p_z	Distributed loading (F/L)
P	Load or axial force (F)
q	Shear flow (F/L)
Q	First moment of area beyond level where shear stress is to be determined (L^3)
R, r	Radius (L)
$S = Z_e$	Section modulus of beam, $S = I/c$ (L^3)
t	Wall thickness (L)
T	Torque or twisting moment (FL)
u, v, w	Displacements in xy, z directions (L)
V	Shear force (F)
γ	Shear strain
Δ	Increment of length (L)
ε	Normal strain
ε_t	Natural strain or true strain
θ	Angle (degree or radian)
ν	Poisson's ratio
σ	Normal stress (F/L^2)
σ_m	Mean stress (F/L^2)
σ_{ys}	Yield stress (F/L^2)
τ	Shear stress (F/L^2)
ϕ	Angular displacement (degree or radian)

3.2 DEFINITIONS AND TYPES OF STRESS

Normally, forces are considered to occur in two forms: surface forces and body forces. *Surface forces* are forces distributed over the surface of the body, such as hydrostatic pressure or the force exerted by one body on another. *Body forces* are

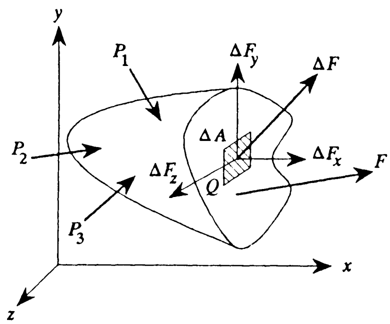


Figure 3-1: Stress.

forces distributed throughout the volume of the body, such as gravitational forces, magnetic forces, or inertial forces for a body in motion. Suppose that a solid is subject to external surface forces P_1, P_2, and P_3 (Fig. 3-1). If the body were cut, a force F would be required to maintain equilibrium. The intensity of this force (i.e., the force per unit area) is defined to be the stress.

The force F will not necessarily be uniformly distributed over the cut. To define the stress at some point Q in a cut perpendicular to the x axis (Fig. 3-1), suppose that the resultant contribution of the internal force F on the area element ΔA at point Q is ΔF, and let the components of ΔF along the x, y, z axes be ΔF_x, ΔF_y, ΔF_z. Stress components are defined as

$$\sigma_x = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A}, \qquad \tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A}, \qquad \tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta F_z}{\Delta A} \tag{3.1}$$

where σ_x is the normal stress and τ_{xy}, τ_{xz} are the shear stresses. Normal stress is the intensity of a force perpendicular to a cut while the shear stresses are parallel to the plane of the element. Tensile stresses are those normal stresses pulling away from the cut, while compressive stresses are those pushing against the cut.

3.3 STRESS COMPONENT ANALYSIS

Sign Convention

An element of infinitesimal dimensions isolated from a solid would expose the stresses shown in Fig. 3-2. The face of an element whose outward normal is directed along the positive direction of a coordinate axis is defined to be a positive face. A negative face has its normal in the opposite direction. Stress components are positive if when acting on a positive face, their corresponding force components are in the positive coordinate direction. Also, stress components are said to be positive when their force components act on a negative face in the negative coordinate direction. Stress components not satisfying these conditions are considered as being negative.

These definitions mean that a normal stress component directed outward from the plane on which it acts (i.e., tension) is positive, while a normal stress directed toward

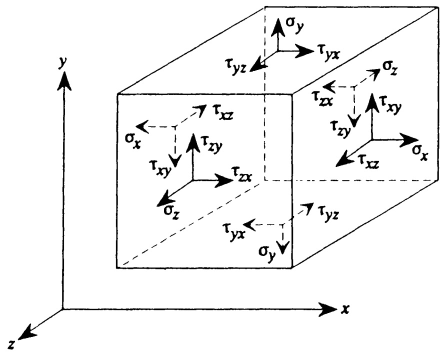


Figure 3-2: Stress components on an element.

the plane on which it acts (i.e., compression) is taken as being negative. Also, a shear stress is positive if the outward normal of the plane on which it acts and the direction of the stress are in coordinate directions of the same sign; otherwise it is negative. The stress components of Fig. 3-2 are positive.

Stress Tensor

In Fig. 3-2, there are three normal stresses components σ_x, σ_y, σ_z, where the single subscript is the axis along which the normal to the cut lies. There are also six shear stress components τ_{xy}, τ_{yx}, τ_{yz}, τ_{zy}, τ_{zx}, τ_{xz}, where the first subscript denotes the axis perpendicular to the plane on which the stress acts and the second provides the direction of the stress component. For example, the shear stress τ_{xy} acts on a plane normal to the x axis and in a direction parallel to the y axis.

The conditions of equilibrium dictate that shear stresses with the same subscripts are equal:

$$\tau_{xy} = \tau_{yx}, \qquad \tau_{xz} = \tau_{zx}, \qquad \tau_{yz} = \tau_{zy} \tag{3.2}$$

In matrix form, the stress components appear as

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \tag{3.3}$$

This state of stress at a point is called a *stress tensor*. The stress tensor is a second-order tensor quantity.

Plane Stress

In the case of plane stress, all stress components (the normal stress and two shear stress components) associated with a given direction are zero. For example, for a thin plate in the yz plane, plane stress corresponds to the x-direction stress components σ_x, τ_{zx}, τ_{yx}, being zero. For the case of plane stress, the state of stress can be determined by three stress components. The stress for thin sheets is usually treated as being in the state of plane stress.

Variation of Normal and Shear Stress in Tension

The bar in Fig. 3-3 is in simple tension. The stresses on planes normal to an axis of the bar are considered to be uniformly distributed and are equal to P/A_0 on cross sections along the length, except near the applied load, where there may be stress concentration (Chapter 6). Here A_0 is the original cross-sectional area of the bar. Consider the stress on an inclined face exposed by passing a plane through the bar at an angle θ, as shown in Fig. 3-4.

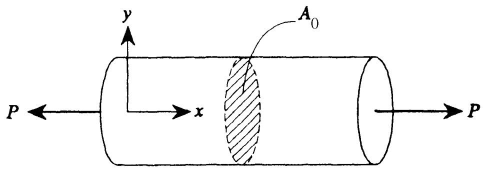


Figure 3-3: Axially loaded bar.

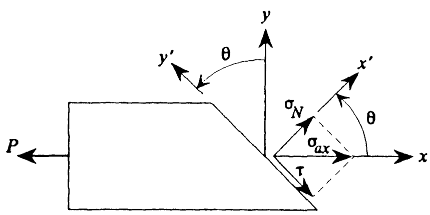


Figure 3-4: Stress on a cross section.

The stress acting in the x direction on the inclined face is $\sigma_{ax} = P/(A_0/\cos\theta)$, where $A_0/\cos\theta$ is the inclined cross-sectional area. This stress can be resolved in terms of the components σ_N and τ as though a force were being resolved since these stresses all act on the same unit of area. These relationships are as follows:

$$\text{normal stress} = \sigma_N = \frac{P\cos\theta}{A_0/\cos\theta} = \frac{P}{A_0}\cos^2\theta \tag{3.4}$$

$$\text{shear stress} = \tau = \frac{P\sin\theta}{A_0/\cos\theta} = \frac{P}{A_0}\sin\theta\cos\theta \tag{3.5}$$

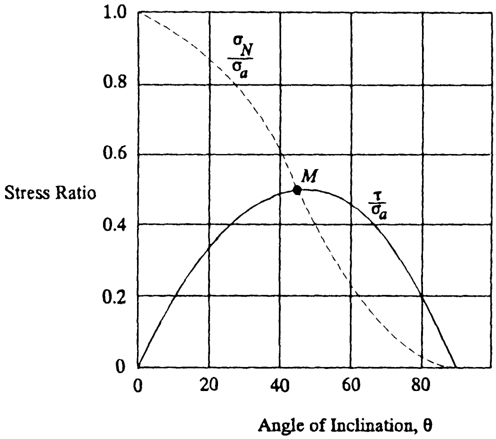


Figure 3-5: Variation of stress with the angle of a plane.

From Eqs. (3.4) and (3.5),

$$\frac{\sigma_N}{P/A_0} = \frac{\sigma_N}{\sigma_a} = \cos^2\theta \tag{3.6}$$

$$\frac{\tau}{P/A_0} = \frac{\tau}{\sigma_a} = \sin\theta\cos\theta \tag{3.7}$$

where σ_a is the axial tensile stress on the section normal to the x axis. Equations (3.6) and (3.7) are plotted in Fig. 3-5. Note that the shear stress is a maximum at 45°, as shown at point M, and that it equals half the maximum tensile stress.

Stress at an Arbitrary Orientation for the Two-Dimensional Case

Consider an element removed from a body subjected to an arbitrary loading in the xy plane (Fig. 3-6a). The stresses σ_x, σ_y, τ_{xy} will occur for the orientation of Fig. 3-6b. Once the state of stress is determined for an element with a particular orientation (such as σ_x, σ_y, τ_{xy} of Fig. 3-6b), the state of stress $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$ at that location for an element in any orientation (Fig. 3-6c, d) can be obtained using the following transformation equations for plane stresses:

$$\sigma_{x'} = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta \tag{3.8a}$$

$$\sigma_{y'} = \tfrac{1}{2}(\sigma_x + \sigma_y) - \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau_{xy}\sin 2\theta \tag{3.8b}$$

$$\tau_{x'y'} = -\tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta \tag{3.8c}$$

Note that it can be found from the equations above that

$$\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y \tag{3.9}$$

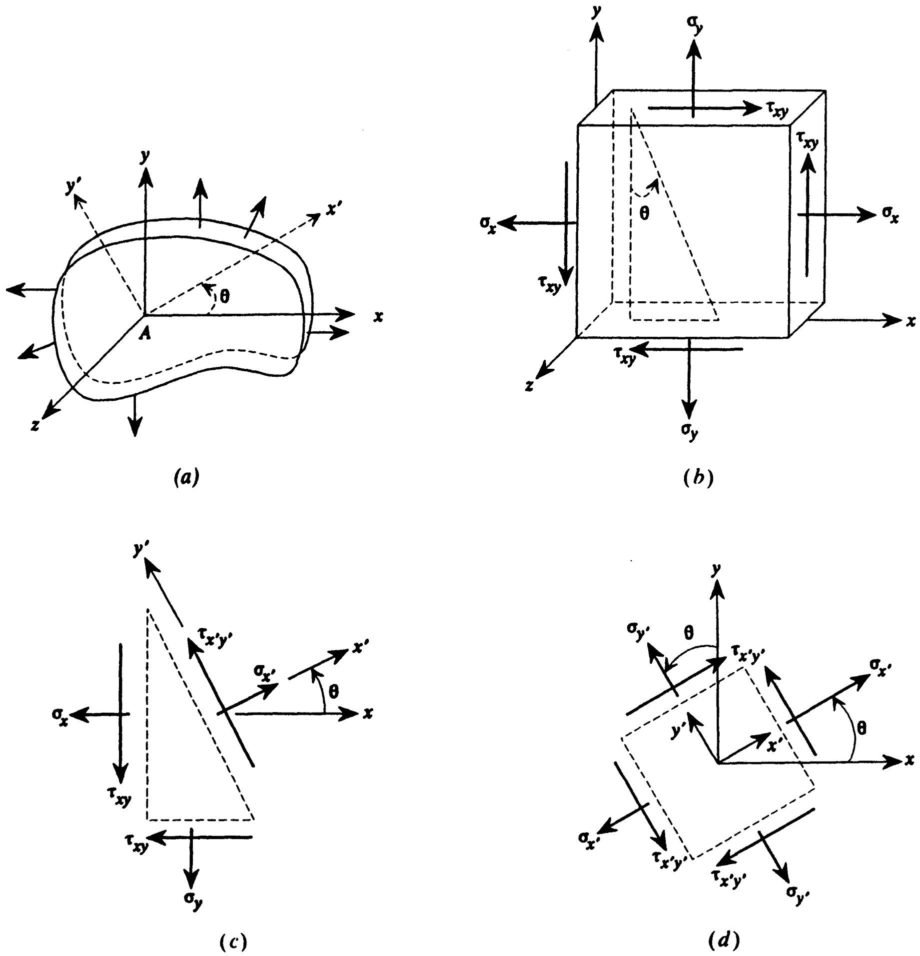


Figure 3-6: (*a*) Object under load; (*b*) element at point *A*; (*c*) element with diagonal at point *A*, taken from (*b*); (*d*) element at point *A* [this can replace the element of (*c*)].

This shows that the sum of the normal stresses is an invariant quantity, independent of the orientation of the element at the point in question.

Example 3.1 State of Stress The state of stress of an element loaded in the xy plane is $\sigma_x = 9000$ psi, $\sigma_y = 3000$ psi, and $\tau_{xy} = 2000$ psi, as shown in Fig. 3-7*a*. Determine the stresses on the element rotated through an angle of 45°.

The state of stress desired can be found by substituting the given values of stresses σ_y, σ_y, τ_{xy} into Eqs. (3.8) with $\theta = 45°$. The results are $\sigma_{x'} = 8000$ psi, $\sigma_{y'} = 4000$ psi, and $\tau_{x'y'} = -3000$ psi. This state of stress is shown in Fig. 3-7*b*.

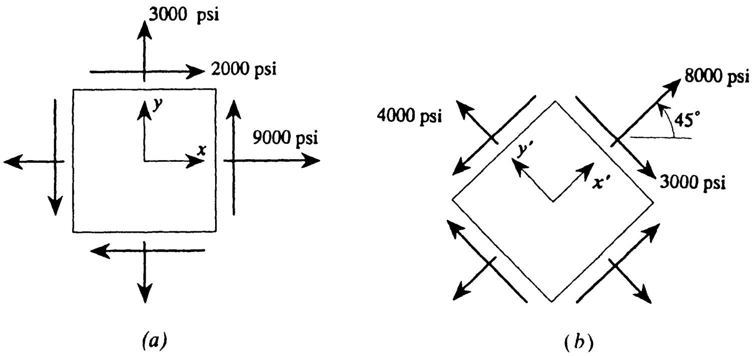


Figure 3-7: Two-dimensional state of stress.

Principal Stresses and Maximum Shear Stress for the Two-Dimensional Case

The maximum value of $\sigma_{x'}$ is found by differentiating Eq. (3.8a) with respect to θ:

$$\frac{d\sigma_{x'}}{d\theta} = 0 = \frac{\sigma_x - \sigma_y}{2}(-2\sin 2\theta) + 2\tau_{xy}\cos 2\theta \tag{3.10}$$

from which

$$\tan 2\theta_1 = 2\tau_{xy}/(\sigma_x - \sigma_y) \tag{3.11}$$

Extreme values of normal stresses occur on the orientations $\theta = \theta_1$ defined by Eq. (3.11). The two values of θ_1 are 90° apart and locate two perpendicular planes of an element (Fig. 3-8). The maximum normal stress occurs on one of the planes while the minimum normal stress occurs on the other.

Principal stresses are defined as the algebraically maximum and minimum values of the normal stresses, and the planes on which they act are called *principal planes*

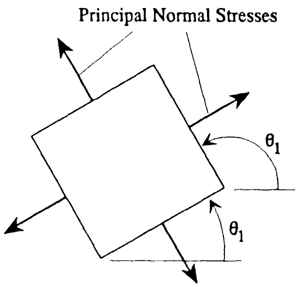


Figure 3-8: Orientation of principal planes.

(Fig. 3-8). From Eq. (3.11), it follows that

$$\sin 2\theta_1 = \frac{\pm\tau_{xy}}{\sqrt{\left[\frac{1}{2}(\sigma_x - \sigma_y)\right]^2 + \tau_{xy}^2}} \tag{3.12a}$$

$$\cos 2\theta_1 = \frac{\pm\frac{1}{2}(\sigma_x - \sigma_y)}{\sqrt{\left[\frac{1}{2}(\sigma_x - \sigma_y)\right]^2 + \tau_{xy}^2}} \tag{3.12b}$$

Substitution of Eqs. (3.12a) and (3.12b) into Eqs. (3.8a) and (3.8b) gives

Algebraic maximum normal stress:
$$\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \sqrt{\frac{1}{4}\left[(\sigma_x - \sigma_y)^2\right] + \tau_{xy}^2} \tag{3.13a}$$

Algebraic minimum normal stress:
$$\sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) - \sqrt{\frac{1}{4}\left[(\sigma_x - \sigma_y)^2\right] + \tau_{xy}^2} \tag{3.13b}$$

Substitution of Eq. (3.11) into Eq. (3.8c) leads to $\tau_{x'y'} = 0$. That is, the shear stress is always zero on the principal planes.

The original stressed element can be used to determine which value of θ_1 for the orientation of principal planes corresponds to σ_1 and which to σ_2. Define the diagonal of a stressed element that passes between the heads of the arrows for the shear stresses as the *shear diagonal*. For example, if τ_{xy} is negative, it should be drawn on the element shown in Fig. 3-9, forming the shear diagonal indicated. Then the direction of σ_1 lies in the 45° arc between the algebraically larger normal stress and the shear diagonal.

To find the maximum shear stress, set $d\tau_{x'y'}/d\theta = 0$ and find that

$$\tau_{\max} = \sqrt{\left[\frac{1}{2}(\sigma_x - \sigma_y)\right]^2 + \tau_{xy}^2} = \frac{1}{2}(\sigma_1 - \sigma_2) \tag{3.14}$$

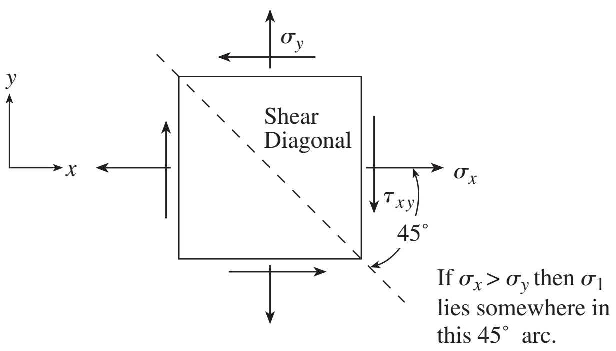


Figure 3-9: Shear diagonal.

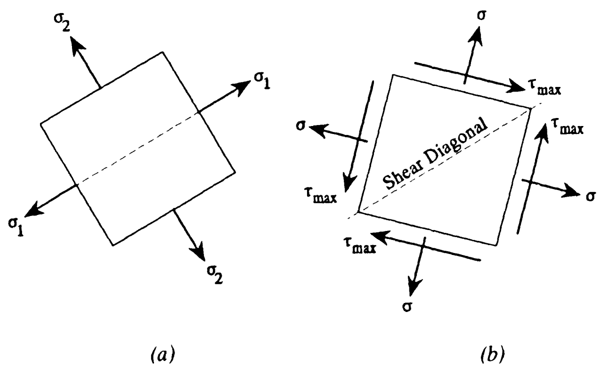


Figure 3-10: Direction of maximum shear stress: (*a*) principal stresses; (*b*) direction of maximum shear stresses for the case of (*a*).

The corresponding values of θ are defined by

$$\tan 2\theta_2 = -\tfrac{1}{2}(\sigma_x - \sigma_y)/\tau_{xy} \tag{3.15}$$

Comparison of Eqs. (3.15) and (3.11) shows that the planes of maximum shear stresses lie 45° away from the planes of the principal stresses.

The fact that the shear diagonal of the element on which the maximum shear stress occurs lies in the direction of the σ_1 stress (Fig. 3-10) assists in determining the proper directions of the maximum shear stresses.

On the planes of maximum shear stress, the normal stress is found by substituting θ_2 of Eq. (3.15) into Eqs. (3.8a) and (3.8b). The normal stress on each plane is

$$\sigma = \tfrac{1}{2}(\sigma_x + \sigma_y) \tag{3.16}$$

Caution must be exercised in using Eq. (3.14) to calculate the maximum shear stress. There is always a third principal stress, σ_3, although it may be equal to zero. When the three principal stresses are considered, as shown later, there are three corresponding shear stresses induced, one of which is the maximum stress.

Example 3.2 Principal Stresses For the element in Fig. 3-7*a*, find the principal stresses and planes and the maximum shear stress.

The principal planes are located by using Eq. (3.11):

$$\tan 2\theta_1 = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{(2)(2000)}{9000 - 3000} = 0.667$$

or $2\theta_1 = 33.7°$ and $180° + 33.7° = 213.7°$. Hence θ_1 is 16.8° and 106.9°. Use of Eqs. (3.13) gives $\sigma_1 = 9605.6$ psi and $\sigma_2 = 2394.4$ psi (Fig. 3-11*a*). The stress σ_1 is located according to the rule for using the shear diagonal.

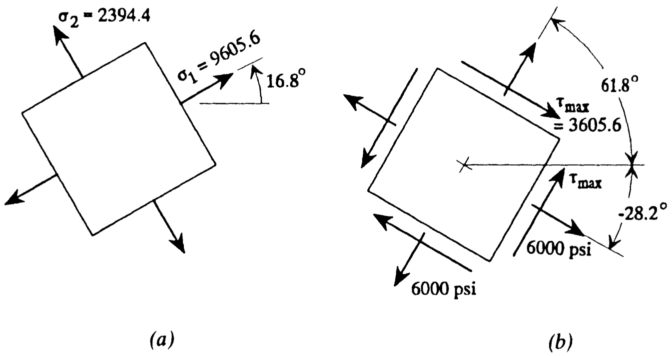


Figure 3-11: (*a*) Principal stress; (*b*) maximum shear stress.

The maximum shear stresses are located on planes identified by θ_2 of Eq. (3.15). Thus $\tan 2\theta_2 = -(9000 - 3000)/(2 \times 2000) = -1.5$, or θ_2 is $-28.2°$ and $61.8°$ (Fig. 3-11*b*). Note that θ_2 can be directly located by the fact that the planes of maximum shear stress are always 45° from the principal planes.

From Eq. (3.14), we obtain $\tau_{\max} = 3605.6$ psi. The corresponding normal stress is, by Eq. (3.16), $\sigma = 6000$ psi (Fig. 3.11*b*).

If $\sigma_3 = 0$, the actual maximum shear stress of the element is

$$\tau_{\max} = (\sigma_1 - \sigma_3)/2 = 9605.6/2 = 4802.8 \text{ psi}$$

Mohr's Circle for a Two-Dimensional State of Stress

A graphical method for representing combined stresses is popularly known as *Mohr's circle method*. As illustrated in Fig. 3-12, the Cartesian coordinate axes represent the normal and shear stresses so that the coordinates σ, τ of each point on the circumference of a circle correspond to the state of stress at an orientation of a stressed element at a point in a body.

Construction of Mohr's Circle

For a known two-dimensional state of stress σ_x, σ_y, and τ_{xy}, Mohr's circle is drawn as follows:

1. On a horizontal axis lay off normal stresses with positive stresses to the right, and on a vertical axis place the shear stresses with positive stresses downward.
2. Find the location of the center of the circle along the σ (horizontal) axis by calculating $\frac{1}{2}(\sigma_x + \sigma_y)$. Tensile stresses are positive; compressive stresses are negative. Plot this point.
3. Plot the point $\sigma = \sigma_x$, $\tau = \tau_{xy}$. Since the positive τ axis is downward, plot a positive τ_{xy} below the σ axis.

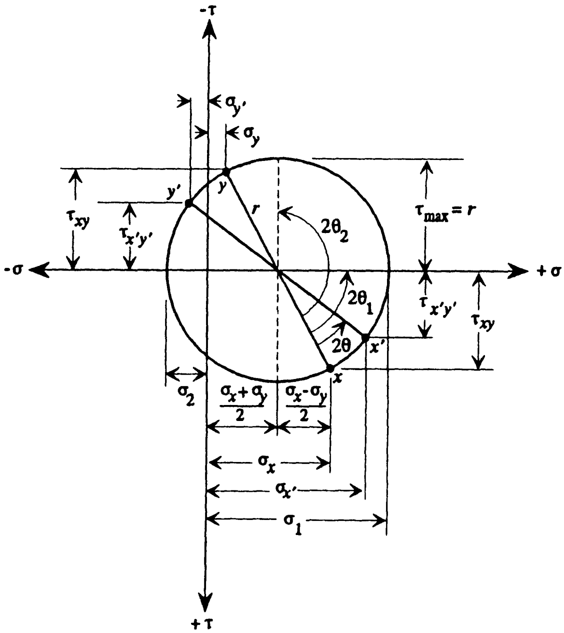


Figure 3-12: Mohr's circle for the two-dimensional stress of Fig. 3-6*b*. This provides the stresses of Fig. 3-6*d* for an orientation of θ.

4. Connect, with a straight line, the center of the circle from step 2 with the point plotted in step 3. This distance is the radius of Mohr's circle. Using $\frac{1}{2}(\sigma_x + \sigma_y)$ on the horizontal axis as the center, draw a circle with the radius just calculated. This is Mohr's circle.

Use of Mohr's Circle

Interpret the coordinates of a point on Mohr's circle as representing the stress components $\sigma_{x'}$ and $\tau_{x'y'}$ that act on a plane perpendicular to the x' axis (Fig. 3-6*d*). The x axis is along the circle radius passing through the plotted point σ_x, τ_{xy}. The angle θ is measured counterclockwise from the x axis. The magnitudes of the angles on Mohr's circle are double those in the physical plane. For example, the stresses $\sigma_{y'}$, $\tau_{x'y'}$, and the y' axis are found on the circle 180° away from $\sigma_{x'}$, $\tau_{x'y'}$ and the x' axis. It should be noted that a special sign convention of shear stress is required to interpret the $\tau_{x'y'}$ associated with $\sigma_{y'}$. That is, positive shear stress is below the σ axis for σ_x while positive shear stress corresponding to σ_y is above the σ axis. From Mohr's circle the following holds:

1. The intersections of the circle with the σ axis are the principal stresses σ_1 and σ_2. These values and their angle of orientation θ relative to the x axis can be

scaled from the diagram or computed from the geometry of the figure. The shear stresses at these two points are zero.

2. The shear stress τ_{max} occurs at the point of greatest ordinate on Mohr's circle. This point has coordinates $\frac{1}{2}(\sigma_x + \sigma_y)$, τ_{max}.
3. The normal and shear stresses on an arbitrary plane for which the normal makes a counterclockwise angle θ with the x axis (Fig. 3-6*d*) are found by measuring a counterclockwise angle 2θ on Mohr's circle from the x axis and then determining the coordinates $\sigma_{x'}$, $\tau_{x'y'}$ of the circle at this angle.

Stress Acting on an Arbitrary Plane in Three-Dimensional Systems

The stress components on planes that are perpendicular to the x, y, z axes are shown in Fig. 3-13, where σ_{Nx}, σ_{Ny}, and σ_{Nz} are stress components on an arbitrary oblique plane P through point 0 of a member. (In the figure the plane P is shown slightly removed from point 0.) The direction cosines of normal N with respect to xy, and z are l, m, and n, respectively.

If the six stress components σ_x, σ_y, σ_z, $\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zy}$, $\tau_{xz} = \tau_{zx}$ at point 0 are known, the stress components on any oblique plane defined by unit normal $N(l, m, n)$ can be computed using

$$\begin{aligned} \sigma_{Nx} &= l\sigma_x + m\tau_{yx} + n\tau_{zx} \\ \sigma_{Ny} &= l\tau_{xy} + m\sigma_y + n\tau_{zy} \\ \sigma_{Nz} &= l\tau_{xz} + m\tau_{yz} + n\sigma_z \end{aligned} \tag{3.17}$$

Normal and Shear Stress on an Oblique Plane

The normal stress σ_N on the plane P is the sum of the projection of the stress components σ_{Nx}, σ_{Ny}, and σ_{Nz} in the direction of normal N. Therefore,

$$\sigma_N = l^2\sigma_x + m^2\sigma_y + n^2\sigma_z + 2mn\tau_{yz} + 2ln\tau_{xz} + 2lm\tau_{xy} \tag{3.18}$$

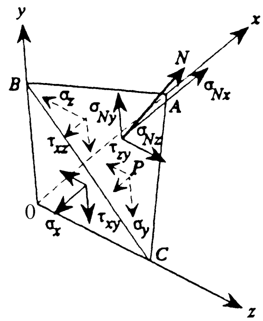


Figure 3-13: Stress components σ_{Nx}, σ_{Ny}, σ_{Nz} on arbitrary plane having normal N.

For a particular plane through point 0, σ_N reaches a maximum value called the *maximum principal stress*. This maximum value along with other principal stresses are the solutions of

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \tag{3.19a}$$

where

$$\begin{aligned} I_1 &= \sigma_x + \sigma_y + \sigma_z \\ I_2 &= \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{xz} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{yz} & \sigma_z \end{vmatrix} \\ &= \sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2 \\ I_3 &= \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{vmatrix} \end{aligned} \tag{3.19b}$$

The quantities I_1, I_2, and I_3 defined in Eq. (3.19b) are invariants of stress and must have the same values for all choices of coordinate axes (x, y, z).

The three roots $(\sigma_1, \sigma_2, \sigma_3)$ of Eq. (3.19a) are the three principal stresses at point 0. The directions of the planes corresponding to the principal stresses, called the *principal planes*, can be obtained from the following linear homogeneous equations in l, m, and n by setting σ in turn equal to σ_1, σ_2, and σ_3 and using the direction cosine relationship $l^2 + m^2 + n^2 = 1$:

$$l(\sigma_x - \sigma) + m\tau_{xy} + n\tau_{xz} = 0, \quad l\tau_{xz} + m\tau_{yz} + n(\sigma_z - \sigma) = 0 \tag{3.20}$$

The magnitude of the shear stress τ_N on plane P is given by

$$\tau_N = \sqrt{\sigma_{Nx}^2 + \sigma_{Ny}^2 + \sigma_{Nz}^2 - \sigma_N^2} \tag{3.21}$$

The maximum value of τ_N at a point in the body plays an important role in certain theories of failure. This shear stress is zero on a principal plane.

Generally speaking, in any stressed body, there are always at least three planes on which the shear stresses are zero; these planes are always mutually perpendicular, and it is on these planes that the principal stresses act.

Maximum Shear Stress in Three-Dimensional Systems

Equations (3.13) and (3.14) deal with two-dimensional systems of stresses. In fact, there are always three principal stresses σ_1, σ_2, σ_3, where σ_3 is the principal stress in the third orthogonal direction. In this three-dimensional situation, three relative maximum shear stresses exist:

$$\tau_1 = \tfrac{1}{2}(\sigma_1 - \sigma_2), \qquad \tau_2 = \tfrac{1}{2}(\sigma_1 - \sigma_3), \qquad \tau_3 = \tfrac{1}{2}(\sigma_2 - \sigma_3) \tag{3.22a}$$

from which the true maximum shear stress can be chosen. This maximum shear stress would be

$$\tau_{\max} = \tfrac{1}{2}(\sigma_{\max} - \sigma_{\min}) \tag{3.22b}$$

This, of course, is the maximum value of τ_N of Eq. (3.21). The three relative maximum shear stresses lie on planes whose normals form 45° angles with the principal stresses involved.

Usually, σ_3 is small or zero in an assumed two-dimensional system of stresses. Then if σ_1 and σ_2 are both positive (in tension), comparison of the magnitudes of the shear stresses in Eqs. (3.22a) indicates that

$$\tau_{\max} = \tfrac{1}{2}(\sigma_1 - \sigma_3) \approx \tfrac{1}{2}\sigma_1 \tag{3.23}$$

would be the true maximum shear stress.

Mohr's Circle for Three Dimensions

Like Mohr's circle for the two-dimensional state of stress, the three mutually perpendicular principal stresses can be represented graphically. Figure 3-14 shows Mohr's circle representation of the triaxial state of stress defined by the three principal stresses in Fig. 3-15. For any section in the σ_1, σ_2 plane (i.e., planes perpendicular to plane 3) there corresponds a circle BA. In the σ_2, σ_3 plane (i.e., planes perpendicular to plane 1) there is a circle CB, and for the σ_3, σ_1 plane there exists a circle CA. From Fig. 3-14, $\sigma_1 = 0A$, $\sigma_2 = 0B$, $\sigma_3 = 0C$, and $\tau_{\max} =$ radius $CA = \frac{1}{2}(\sigma_1 - \sigma_3)$.

It can be shown [3.1] that all possible stress conditions for the body fall within the shaded area between the circles in Fig. 3-14.

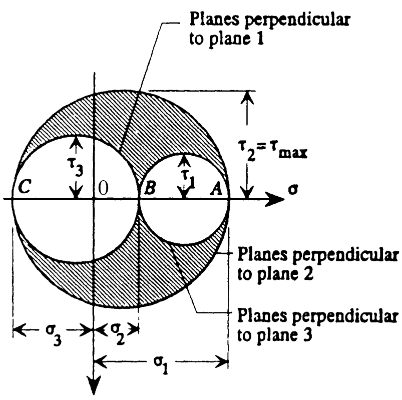


Figure 3-14: Mohr's circle for a three-dimensional state of stress.

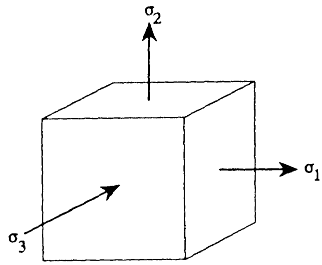


Figure 3-15: Triaxial state of stress.

Mohr's circles for some common states of stress are given in Table 3-1.

Example 3.3 Mohr's Circle For the state of stress shown in Fig. 3-16*a*, using Mohr's circle, determine graphically (a) the stress components on the element rotated through an angle of 45°, (b) the principal stresses and planes, and (c) the maximum shear stresses.

First, find the center 0′ of Mohr's circle on the σ axis by using $\sigma = \frac{1}{2}(\sigma_x + \sigma_y) =$ 6000 psi, and plot the point Q with coordinates $(\sigma, \tau) = (\sigma_x, \tau_{xy}) = (9000, 2000)$. Then draw a circle with radius equal to the distance between these two points, $0'Q$. This is measured (or calculated) to be 3605.6 psi.

(a) The stress components on the element rotated through an angle of 45° are represented on Mohr's circle by rotating $0'Q$ counterclockwise $2\theta = 2 \times 45 = 90°$.

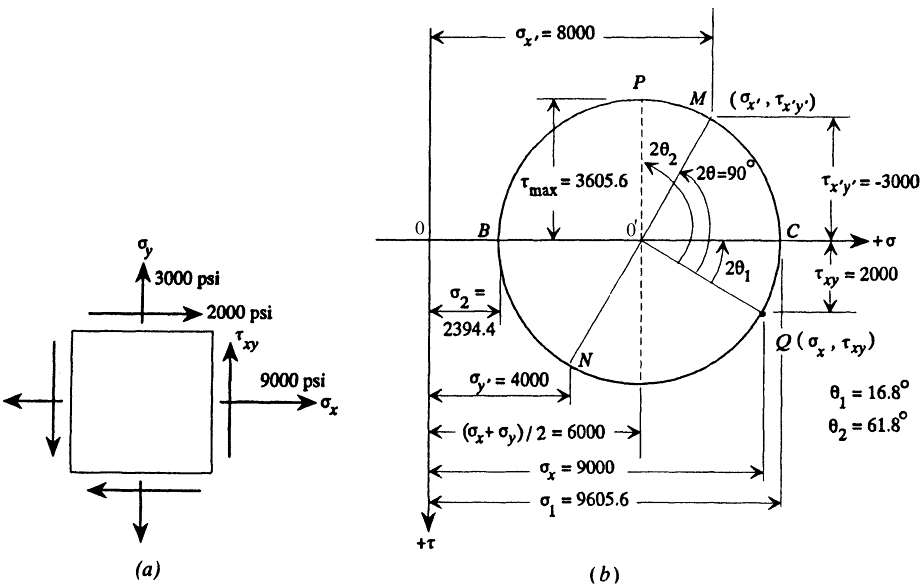


Figure 3-16: Example of Mohr's circle: (*a*) state of stress; (*b*) stress components on Mohr's circle.

This identifies the x' axis. The intersection M of the x' axis (i.e., $0'M$) with the circle gives $\sigma_{x'} = 8000$ psi and $\tau_{x'y'} = -3000$ psi. The $\sigma_{y'}$ stress, which is found 180° away from the x' axis ($0'M$), is 4000 psi. Refer to Fig. 3-16*b*.

(b) $\sigma_1 = 0C = 00' + 0'C = 6000 + 3605.6 = 9605.6$ psi, $2\theta_1 = 33.6°$ or $\theta_1 = 16.8°$

$$\sigma_2 = 0B = 00' - 0'B = 6000 - 3605.6 = 2394.4 \text{ psi}$$

$$\theta_1 = 90° + 16.8° = 106.8°$$

$$\sigma_3 = 0$$

(c) For a section in the σ_1, σ_2 plane, the maximum shear stresses occur on the vertical through the center of the circle (i.e., $0'P$). We measure $0'P = \tau_{max} = 3605.6$ psi and $2\theta_2 = 123.6°$ or $\theta_2 = 61.8°$. But since $\sigma_3 = 0$, the actual maximum shear stress of the element is $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) = 4802.3$ psi.

Octahedral Stress

Suppose that coordinate axes x, y, z are principal axes that are perpendicular to each of the principal planes, respectively. In three dimensions there are eight planes (the *octahedral* planes) that make equal angles with respect to the x, y, z directions; that is, the absolute values of the direction cosines of the eight planes are equal, $|l| = |m| = |n| = \frac{1}{3}\sqrt{3}$. The normal and shear stress components associated with each of these planes are called the *octahedral normal stress* σ_{oct} and the *octahedral shear stress* τ_{oct}.

For this case Eqs. (3.17) and (3.18) become

$$\sigma_{Nx} = \tfrac{1}{3}\sqrt{3}\sigma_1, \qquad \sigma_{Ny} = \tfrac{1}{3}\sqrt{3}\sigma_2, \qquad \sigma_{Nz} = \tfrac{1}{3}\sqrt{3}\sigma_3$$

and

$$\sigma_{oct} = \sigma_N = \tfrac{1}{3}\sigma_1 + \tfrac{1}{3}\sigma_2 + \tfrac{1}{3}\sigma_3 = \tfrac{1}{3}I_1 \tag{3.24a}$$

Substituting $\sigma_{Nx}, \sigma_{Ny}, \sigma_{Nz}$, and σ_N into Eq. (3.21) yields

$$\begin{aligned}\tau_{oct} = \tau_N &= \tfrac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2\right]^{1/2} \\ &= \tfrac{1}{3}(2I_1^2 - 6I_2)^{1/2}\end{aligned} \tag{3.24b}$$

In general, $\sigma_x, \sigma_y, \sigma_z$ are not principal stresses and $\tau_{xy}, \tau_{yz}, \tau_{zx}$ are not zero. However, the quantities I_1, I_2, and I_3 are invariant. The quantities σ_{oct} and τ_{oct} become

$$\sigma_{oct} = \tfrac{1}{3}I_1 = \tfrac{1}{3}(\sigma_x + \sigma_y + \sigma_z) \tag{3.25a}$$

$$\begin{aligned}\tau_{oct} &= \tfrac{1}{3}(2I_1^2 - 6I_2)^{1/2} \\ &= \tfrac{1}{3}\left[(\sigma_x - \sigma_y)^2 + (\sigma_x - \sigma_z)^2 + (\sigma_y - \sigma_z)^2 + 6(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2)\right]^{1/2}\end{aligned} \tag{3.25b}$$

Mean and Deviator Stress

The mean stress σ_m is defined by

$$\sigma_m = \tfrac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \tfrac{1}{3}I_1 \tag{3.26}$$

It is often contended that yielding and plastic deformation of some metals are basically independent of the applied normal mean stress σ_m. As a consequence, it is useful to separate σ_m from the other stresses so that the stress tensor [Eq. (3.3)] is expressed in terms of the mean and deviator stress

$$T = T_m + T_d \tag{3.27a}$$

where

$$T = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix},$$

$$T_m = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$$

and

$$T_d = \begin{bmatrix} \frac{1}{3}(2\sigma_x - \sigma_y - \sigma_z) & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \frac{1}{3}(2\sigma_y - \sigma_x - \sigma_z) & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \frac{1}{3}(2\sigma_z - \sigma_y - \sigma_x) \end{bmatrix}$$

$$= \begin{bmatrix} S_x & S_{xy} & S_{xz} \\ S_{yx} & S_y & S_{yz} \\ S_{zx} & S_{zy} & S_z \end{bmatrix} \tag{3.27b}$$

The matrix T_m is referred to as the *mean stress tensor* and the matrix T_d the *deviator stress tensor*. The components S_{ij} of T_d are called the *deviator stresses*. For stress tensor T, the invariants of stress, I_1, I_2, and I_3, are defined in Eq. (3.19b). Similarly, for tensors T_m, T_d, the quantities I_{1m}, I_{1d}, I_{2m}, I_{2d}, and I_{3m}, I_{3d} can also be defined. The stress invariants for principal axes x, y, z are as follows:

$$I_{1m} = I_1 = 3\sigma_m, \quad I_{2m} = \tfrac{1}{3}I_1^2 = 3\sigma_m^2, \quad I_{3m} = \tfrac{1}{27}I_1^3 = \sigma_m^3 \quad (\text{for } T_m) \tag{3.28a}$$

$$\left.\begin{aligned} I_{1d} &= 0 \\ I_{2d} &= I_2 - \tfrac{1}{3}I_1^2 = \left(-\tfrac{1}{6}\right)\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \\ I_{3d} &= I_3 - \tfrac{1}{3}I_1 I_2 + \tfrac{2}{27}I_1^3 \\ &= \tfrac{1}{27}(2\sigma_1 - \sigma_2 - \sigma_3)(2\sigma_2 - \sigma_3 - \sigma_1)(2\sigma_3 - \sigma_1 - \sigma_2) \end{aligned}\right\} \quad (\text{for } T_d)$$

$$\tag{3.28b}$$

The principal values of the deviator stresses are

$$\begin{aligned} S_1 &= \sigma_1 - \sigma_m = \tfrac{1}{3}\left[(\sigma_1 - \sigma_3) + (\sigma_1 - \sigma_2)\right] \\ S_2 &= \sigma_2 - \sigma_m = \tfrac{1}{3}\left[(\sigma_2 - \sigma_3) + (\sigma_2 - \sigma_1)\right] \\ S_3 &= \sigma_3 - \sigma_m = \tfrac{1}{3}\left[(\sigma_3 - \sigma_1) + (\sigma_3 - \sigma_2)\right] \end{aligned} \tag{3.29a}$$

It is apparent that

$$S_1 + S_2 + S_3 = 0 \tag{3.29b}$$

The deviator stresses are sometimes used in theories of failure and in the theory of plasticity.

3.4 RELATIONSHIP BETWEEN STRESS AND INTERNAL FORCES

Both stress components and internal-force components are used to describe the state of the internal action of a solid. They are related in the sense that the internal forces are the resultant or total stresses. These are often referred to as *stress resultants*. Comparison of Fig. 3-17*a* and *b* for a bar cut perpendicular to the x axis leads to the following relationships:

$$F_x = P = \int_A \sigma_x \, dA \tag{3.30a}$$

$$V_y = \int_A \tau_{xy} \, dA \tag{3.30b}$$

$$V = V_z = \int_A \tau_{xz} \, dA \tag{3.30c}$$

$$M_x = T = \int_A \tau_{xz} y \, dA - \int_A \tau_{xy} z \, dA \tag{3.30d}$$

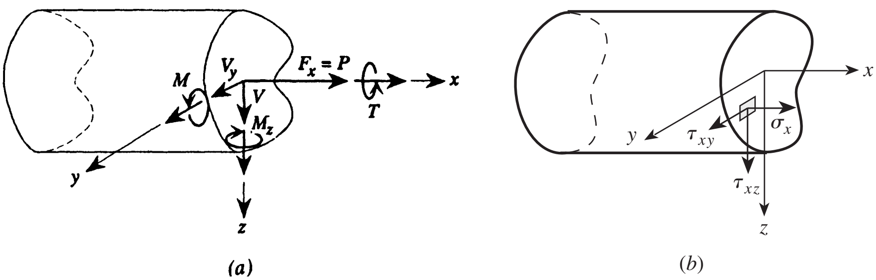


Figure 3-17: Internal forces and stresses.

$$M = M_y = \int_A \sigma_x z \, dA \tag{3.30e}$$

$$M_z = -\int_A \sigma_x y \, dA \tag{3.30f}$$

Average Shear Stress

The force acting on a plane cut in a body is called a *shear force*. Often an approximation for the stress acting on the plane is obtained by dividing the shear force by the area over which it acts. Thus,

$$\tau = \frac{\text{force}}{\text{area}} = \frac{V}{A} \tag{3.31}$$

where τ is the shear stress, V the total force acting across and parallel to a cut plane, and A the cross-sectional area for the cut. This approximation, which is based on the assumption of a uniform distribution of stress, is called the *average shear stress*.

3.5 DIFFERENTIAL EQUATIONS OF EQUILIBRIUM

For equilibrium to exist throughout a solid for two-dimensional problems, the following differential equations must be satisfied:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + p_x = 0, \qquad \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + p_y = 0 \tag{3.32a}$$

In the case of three-dimensional stress, the equations above become

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + p_x = 0 \tag{3.32b}$$

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + p_y = 0 \tag{3.32c}$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + p_z = 0 \tag{3.32d}$$

where p_x, p_y, and p_z represent body forces per unit volume, such as those generated by weight or magnetic effects.

3.6 ALLOWABLE STRESS

Either in analyzing an existing structure or in designing a new structure, it is very important to know what constitutes a "safe" stress level. The ability of a member to resist failure is limited to a certain level. A prescribed stress level that is not to be exceeded when a member is subjected to the expected load is the *allowable* or *working stress*. The allowable stress is sometimes based on the stress level at the transition between elastic and nonelastic material behavior (i.e., yield stress). It may also be based on the occurrence of fracture (rupture) or the highest or ultimate stress that can occur in a member. In most cases the allowable stress is calculated to be lower than the yield or ultimate stress, the reduction being determined by a factor of safety. Values of allowable stress are established by local and federal agencies and by technical organizations such as the American Society of Mechanical Engineers (ASME).

3.7 RESIDUAL STRESS

Residual stress (or *lockup stress*, *initial stress*) [3.3–3.7] is defined as that stress that is internal or locked into a part or assembly even though the part or assembly is free from external forces or thermal gradients. Such residual stress, whether in an individual part or in an assembly of parts, can result from a mismatch or misfit between adjacent regions of the same part or assembly.

It is often important to consider residual stresses in failure analysis and design, although residual stresses tend to be difficult to visualize, measure, and calculate [3.8]. Residual stresses are three-dimensional, self-balanced systems that need not be harmful. In fact, it may be desirable to have high compressive residual stress at the surface of parts subject to fatigue or stress corrosion.

3.8 DEFINITION OF STRAIN

Strain can be defined in terms of normal and shear strain. *Normal strain* is defined as the change in length per unit length of a line segment in the direction under consideration. Normal strain is a dimensionless quantity denoted by ε_i, where the subscript i indicates the direction. Normal strain is taken as positive when the line segment elongates and negative when the line segment contracts. For the member in Fig. 3-18 with uniaxial stress,

$$\varepsilon_x = \frac{2\Delta}{2L} = \frac{\Delta}{L} = \frac{L_f - L}{L}, \qquad \varepsilon_y = -\frac{2\Delta h}{2h} = -\frac{\Delta h}{h} = \frac{h_f - h}{h} \tag{3.33}$$

where $2L$ and $2h$ are the original dimensions and $2L_f$ and $2h_f$ are the postdeformation dimensions.

Shear strain is defined as the tangent of the change in angle of a right angle in a member undergoing deformation. It is a dimensionless quantity. The symbol for the strain is γ_{ij}, where the subscripts have meanings similar to the subscripts for shear stress. For the small shear strains encountered in most engineering practice (usually less than 0.001), the tangent of the change in angle is very nearly equal to the angle change in radians. Positive shear strains are associated with positive shear

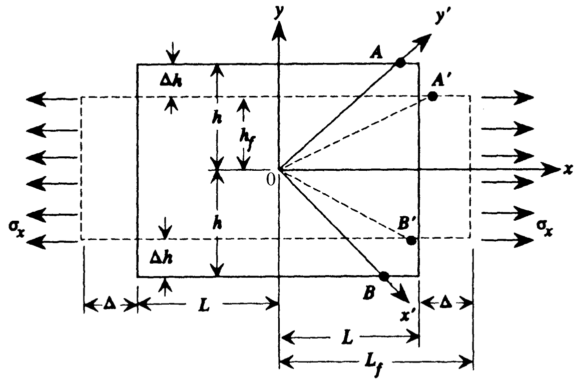


Figure 3-18: Elongation of an element.

stresses (Fig. 3-19*a*); negative shear strains correspond to negative shear stresses (Fig. 3-19*b*). Refer to the x', y' axes of Fig. 3-18. If this member is lengthened and thinned, A and B will move to new positions A' and B'. Angle $A'0B'$ is now less than 90°. The tangent of the total change in angle is the *shear strain*.

Another useful definition of strain is the change in length divided by the instantaneous value of the length (rather than the original length):

$$\varepsilon_t = \int_L^{L_f} \frac{d\ell}{\ell} = \ln \frac{L_f}{L} \tag{3.34}$$

where ε_t is referred to as the *natural* (or *true*) *strain*. The concept of true strain is very useful in handling problems in plasticity and metal forming. For the very small strains for which the equations of linear elasticity are valid, the two types of strains (strain and true strain) give almost the same values.

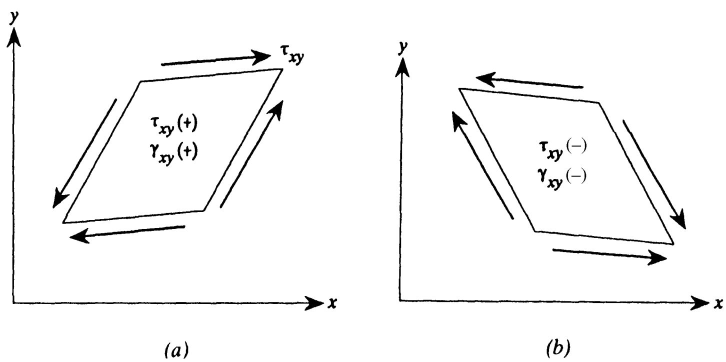


Figure 3-19: Shear strain sign.

3.9 RELATIONSHIP BETWEEN STRAIN AND DISPLACEMENT

In general, the state of strain at a point in a body is determined by six strains, ε_x, ε_y, ε_z, γ_{yx}, γ_{xz}, and γ_{yz}, arranged in the same fashion as stresses. These components can be assembled into a strain tensor similar to the stress tensor.

If u, v, and w are three displacement components at a point in a body for the xy, and z directions of coordinate axes, small strains are related to the displacements through the geometric relationships

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x}, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{yx} \\ \varepsilon_y &= \frac{\partial v}{\partial y}, \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \gamma_{zx} \\ \varepsilon_z &= \frac{\partial w}{\partial z}, \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \gamma_{zy} \end{aligned} \tag{3.35}$$

In the case of *plane strain* (zero strains in the z direction, i.e., $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$), the foregoing equations become

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_y = \frac{\partial v}{\partial y}, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{yx} \tag{3.36}$$

It can be shown that to assure unique continuous displacements, the strains cannot be independent. For example, the *compatibility condition*

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{3.37}$$

must hold. That is, the three strains of Eq. (3.36) must satisfy Eq. (3.37) to assure that the two displacements u, v are single valued and continuous.

3.10 ANALYSIS OF STRAIN

The strain components possess the same sort of tensor characteristics as the stress components. Hence, strains follow the same rules as stresses when axes are rotated. There are principal axes for strain, and a Mohr's circle for strain can be used to evaluate strain components at various orientations. The only difference is that the vertical axis is $\frac{1}{2}\gamma$ rather than τ, which is used with Mohr's circle of stress. Therefore, the normal strain ε_N at a point in the direction of N that makes a counterclockwise angle θ_N with the x axis is

$$\varepsilon_N = \varepsilon_x \cos^2 \theta_N + \varepsilon_y \sin^2 \theta_N + \gamma_{xy} \sin \theta_N \cos \theta_N \tag{3.38}$$

In strain measurement, the majority of problems are two-dimensional. The extensions (or normal strain) in one or more directions are the quantities most often measured.

3.11 ELASTIC STRESS–STRAIN RELATIONS

Poisson's Ratio

For a bar of elastic material having the same mechanical properties in all directions and under a condition of uniaxial loading, measurements indicate that the lateral compressive strain is a fixed fraction of the longitudinal extensional strain. This fraction is known as *Poisson's ratio* ν. In the case of the member of Fig. 3-18,

$$\varepsilon_y = -\nu\varepsilon_x \tag{3.39}$$

Like the modulus of elasticity E of the following paragraph, Poisson's ratio is a material constant that can be determined experimentally. For metals it is usually between 0.25 and 0.35. It can be as low as 0.1 for certain concretes and as high as 0.5 for rubber.

Hooke's Law

The stresses and strains are related to each other by the properties of the material. Equations of this nature are known as *material laws* or, in the case of elastic solids, as *Hooke's law*. For a three-dimensional state of stress and strain, Hooke's law for isotropic material appears as

$$\begin{aligned} \varepsilon_x &= (1/E)[\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \varepsilon_y &= (1/E)[\sigma_y - \nu(\sigma_x + \sigma_z)] \\ \varepsilon_z &= (1/E)[\sigma_z - \nu(\sigma_x + \sigma_y)] \\ \tau_{ij} &= G\gamma_{ij} \qquad (i, j = x, y, z; i \neq j) \end{aligned} \tag{3.40}$$

where E is the *modulus of elasticity*, ν is Poisson's ratio, and G is the *shear modulus*. The dimensions of G and E are force per unit area [e.g., lb/in^2 or N/m^2 (Pa)]. Typical values of E and ν for some materials are listed in Table 4-3. The *bulk modulus K* (also called *volumetric modulus of elasticity*, *modulus of dilation*, *modulus of volume expansion*, or *modulus of compressibility*) is a material constant defined as the ratio of the hydrostatic stress $\sigma_1 = \sigma_2 = \sigma_3$ (shear stresses are zero) to the volumetric strain (change in volume divided by the original volume). Of the many different material constants (e.g., E, ν, G, and K), only two are independent if the material is isotropic. Table 3-2 lists the relationships between commonly used material constants.

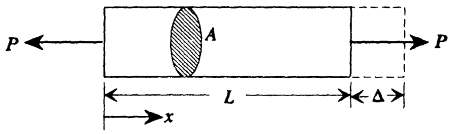


Figure 3-20: Extension.

3.12 STRESS AND STRAIN IN SIMPLE CONFIGURATIONS

Direct Axial Loading (Extension and Compression)

A typical tension member is shown in Fig. 3-20. It is assumed that the force acts uniformly over the cross section so that the stress at any point is

$$\sigma_x = P/A \tag{3.41}$$

As a result of the force P, the bar elongates an amount Δ. In terms of strain ε_x along the bar,

$$\varepsilon_x = \Delta/L \tag{3.42}$$

The quantities σ_x and ε_x are called *engineering stress* and *strain* since they are based on the original dimensions of the bar.

Using Hooke's law for the axial fibers, $\sigma_x = E\varepsilon_x$, Eq. (3.41) becomes

$$\varepsilon_x = P/EA \tag{3.43}$$

or

$$\Delta = PL/AE \tag{3.44}$$

Frequently, it is convenient to relate the extension of a bar to the extension of a spring. If the force in the spring of Fig. 3-21a is linearly proportional to its displace-

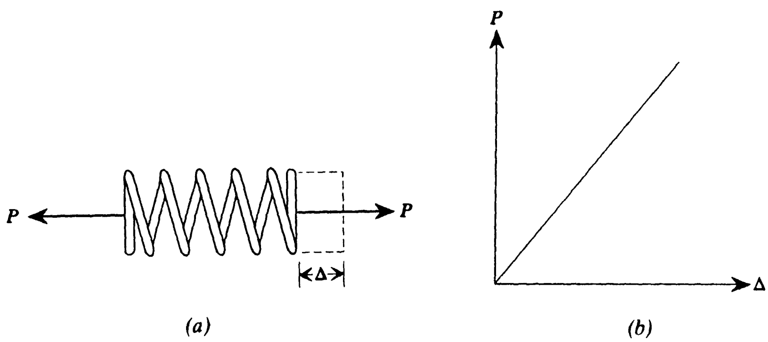


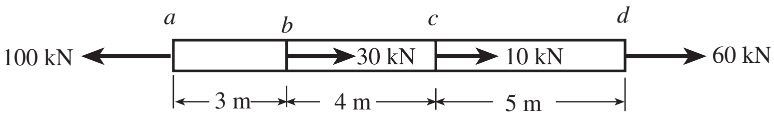
Figure 3-21: Spring.

ment (Fig. 3-21*b*), the constant of proportionality is the *spring constant*

$$k = P/\Delta \tag{3.45}$$

The constant k is also referred to as the *stiffness coefficient*. The reciprocal of the stiffness coefficient, $1/k$, is the *flexibility coefficient*.

Example 3.4 Elongation of a Bar A steel bar with a uniform cross section of 1000 mm^2 is subject to the uniaxial forces shown in Fig. 3-22*a*. Calculate the total elongation of the bar ($E = 200$ GN/m^2).



(*a*)

(*b*)

Figure 3-22: Bar.

The entire bar is in equilibrium since the sum of the axial forces is zero. The total elongation is determined by separating the bar into three sections, finding the elongation of each, and adding these elongations. The conditions of equilibrium give the internal force in each section. Thus, for Fig. 3-22*b*, $\sum F_H = 0$: $-F_{bc} + 10 + 60 = 0$ or the internal force $F_{bc} = 70$ kN in tension. Similar manipulations give $F_{ab} = 100$ kN, $F_{cd} = 60$ kN, both in tension. Then from Eq. (3.44),

$$\begin{aligned}\Delta = \Delta_{ab} + \Delta_{bc} + \Delta_{cd} &= \frac{1}{AE}[(FL)_{ab} + (FL)_{bc} + (FL)_{cd}] \\ &= \frac{(100\text{ kN})(3\text{ m}) + (70\text{ kN})(4\text{ m}) + (60\text{ kN})(5\text{ m})}{(1000\text{ mm}^2)(200\text{ GN/m}^2)} \\ &= 4.4 \times 10^{-3}\text{ m} = 4.4\text{ mm}\end{aligned} \tag{1}$$

There are some differences between compression and tension. First, in compression, instability failure by buckling may occur depending on the geometry, especially the length. Second, for ductile materials, there is no apparent ultimate strength in compression.

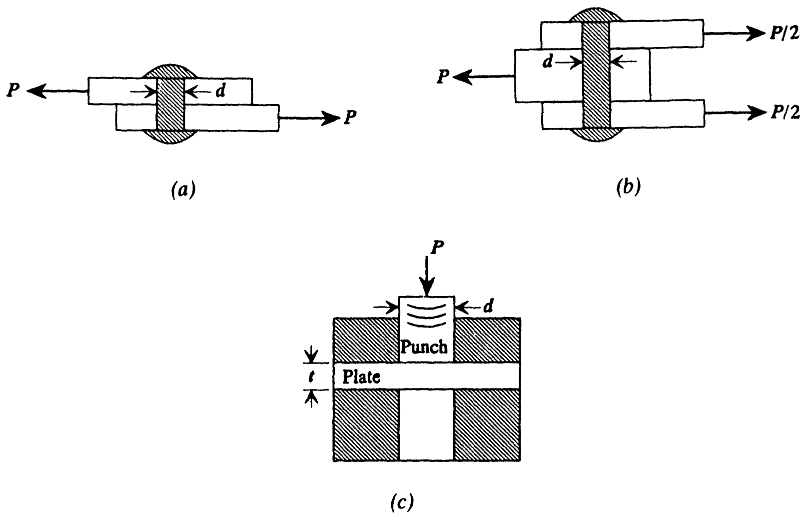


Figure 3-23: Examples of shear: (*a*) single shear; (*b*) double shear; (*c*) punch on a plate.

Direct Shear in Connections

Shear may be considered to be a process whereby parallel planes move relative to one another. In direct shear, the shear stress can be calculated as an average stress. Some examples of direct shear are shown in Fig. 3-23. For the configurations in Fig. 3-23*a* and *b*, the shear stresses in the bolts of cross-sectional area A are

$$\text{Single stress:} \qquad \tau = \frac{P}{A} = \frac{P}{\pi d^2/4} = \frac{4P}{\pi d^2} \tag{3.46a}$$

$$\text{Double stress:} \qquad \tau = \frac{P}{2A} = \frac{P}{2\pi d^2/4} = \frac{2P}{\pi d^2} \tag{3.46b}$$

The direct shear in Fig. 3-23*c* occurs as a punch tries to penetrate a plate. If the punch diameter is d and the plate thickness is t, the shear stress τ in the plate is

$$\tau = P/A = P/(\pi\, dt) \tag{3.46c}$$

Torsion

For a bar subject to an applied torque (Fig. 3-24), the torsional or shear stresses τ on a cross section of circular shape, either solid or hollow, are linearly proportional in magnitude to the distance r from the centroidal axis of the bar. This stress, which acts normal to the radius, is given by

$$\tau = Tr/J \tag{3.47}$$

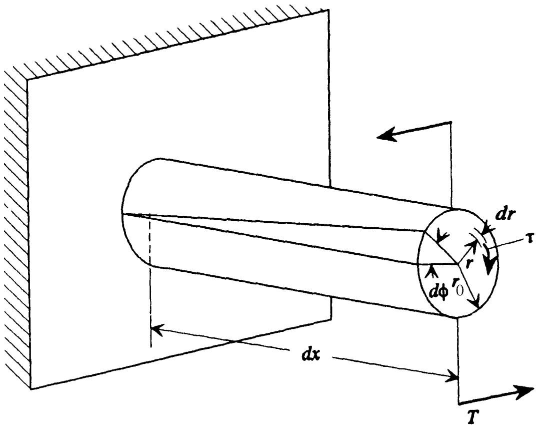


Figure 3-24: Torsion.

where τ is the shear stress [force per unit area, psi or N/M^2 (Pa)], T is the torque or twisting moment (length $\times$ force, in.-lb or $N \cdot m$), r is the radial distance from longitudinal axis (length, in. or m), and J is the torsional constant (length to the fourth power, in^4 or mm^4) of cross section; if the cross-sectional shape is circular, $J = I_x$, the polar moment of inertia about the longitudinal axis.

It can be seen from Eq. (3.47) that the highest stresses occur in the outer edge fibers:

$$\tau_{max} = Tr_0/J \tag{3.48}$$

where r_0 is the radial distance to the outer boundary of the circular cross section.

The shear strain γ for any section of the bar is given by

$$\gamma = \tau/G = Tr/GJ \tag{3.49}$$

In addition, since at any distance dx from the fixed end of the bar, $\gamma = r\, d\varphi/dx$, Eq. (3.49) shows that

$$\frac{d\phi}{dx} = T/GJ \tag{3.50a}$$

which upon integration gives

$$\phi = TL/GJ \tag{3.50b}$$

Torsion of Thin-Walled Shafts and Tubes of Circular Cross Sections

For the thin-walled circular section of Fig. 3-25, if the shear stress is assumed to be uniformly distributed across the thickness, the equilibrium conditions give

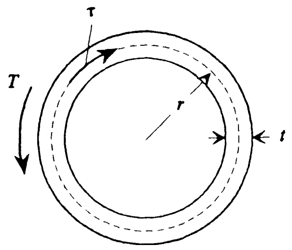


Figure 3-25: Thin-walled torsion.

$$T = 2\pi r^2 t\tau, \qquad \tau = T/2\pi r^2 t \tag{3.51a}$$

where r is the radius to the midwall.

Since the torsional constant for a thin circular section is approximately $J = 2\pi r^3 t$, the shear stress can be written as

$$\tau = Tr/J \tag{3.51b}$$

Equation (3.51a) also follows directly from Eq. (3.47). The angle of twist of this thin-walled circular section is still given by Eq. (3.50b).

Torsion of Thin-Walled Noncircular Tubes For thin-walled noncircular sections it is assumed that the wall thickness is small compared to the overall dimensions of the cross section and that the stress is uniform through the wall thickness. Experiments and comparisons with more exact analyses have shown this latter assumption to be reasonable for most thin-walled sections in the elastic range.

The formulas for thin-walled tubes (Fig. 3-26) are

$$q = T/2A^* \tag{3.52a}$$

$$\phi = \frac{TL}{GJ} \quad \text{or} \quad \frac{d\phi}{dx} = \frac{T}{GJ} \tag{3.52b}$$

$$q = \tau t \tag{3.52c}$$

where q is the shear flow, A^* is the area enclosed by the middle line of the wall, and J is the torsional constant.

For constant t, Eq. (3.52b) becomes

$$\frac{d\phi}{dx} = \frac{\tau S}{2A^* G} = \frac{TS}{4A^{*2} Gt} \tag{3.53}$$

where S is the total length of the middle line of the wall of the cross section.

Note that although the shear flow q from Eq. (3.52a) is constant around the wall, the shear stress $\tau = q/t$ of Eq. (3.52c) can vary with t. The largest shear stress occurs where the wall is thinnest, and vice versa. Also note that no distinction is made by Eq. (3.52a) between different cross-sectional shapes. According to this formula, all

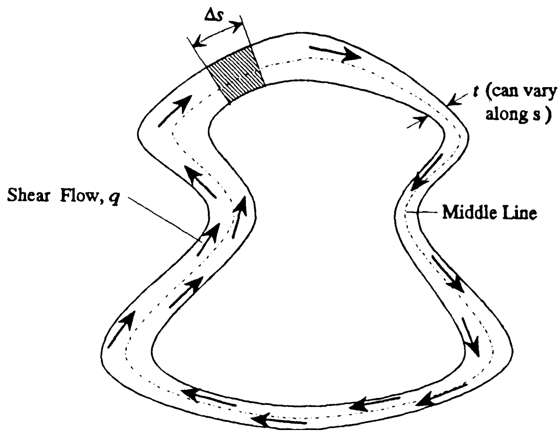


Figure 3-26: Thin-walled tube.

cross-sectional geometries with the same enclosed area A^* will experience the same shear flow for the same torque T. Equations (3.52) and (3.53) are simple to apply and quite accurate for thin-walled closed sections of arbitrary cross-sectional shape. Chapters 2 and 12 provide formulas for various cross-sectional shapes, including multicell thin-walled beams.

If the walls of the hollow shaft are very thin, the possibility of buckling should be considered. Thus, a shaft safe from the standpoint of yield stress level may well be unstable.

A Useful Relation between Power, Speed of Rotation, and Torque

Power is the measure of work developed per unit time. The work done by a torque T during one revolution of a shaft is $2\pi T$. For a shaft rotating at n revolutions per minute (rpm), the work done per minute is $2\pi Tn$. In the U.S. Customary System, the usual unit of power is foot-pounds per second. In engineering work, a larger unit called *horsepower* (hp) is often used:

$$1 \text{ hp} = 33{,}000 \text{ ft-lb/min} \tag{3.54a}$$

If T is in inch-pounds, the horsepower transmitted is

$$\text{hp} = \frac{2\pi Tn}{12(33{,}000)} = \frac{Tn}{63{,}000} \tag{3.54b}$$

For the International System (SI), the unit of power is the watt, $\text{W} = \text{N} \cdot \text{m/s}$. If T is in newton-meters,

$$\text{W} = \frac{2\pi Tn}{60} \tag{3.54c}$$

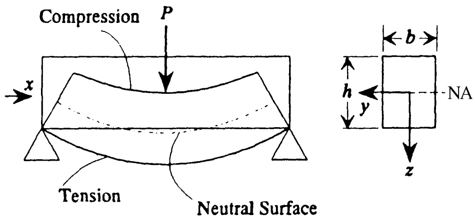


Figure 3-27: Beam under loading.

Normal and Shear Stress of Beams

When a simple beam bends under vertical downward load, the top fibers shorten the most and the bottom fibers lengthen the most (Fig. 3-27). Between the top and the bottom fibers, there exists a layer or surface that remains neutral; neither tension nor compression is generated in it, although it is curved like the rest of the layers. Hence, this layer is called the *neutral surface*. It is assumed that the fiber deformations are directly proportional to the distance from the neutral surface. This fundamental assumption about the geometry of deformation of a beam is stated as follows: *Plane sections normal to the axis of a beam remain plane as the beam is bent.*

The intersection of a cross-sectional plane with the neutral surface is called the *neutral axis* (NA). For example, the y axis shown in Fig. 3-27 is the neutral axis of the cross section. It can be shown that the neutral axis passes through the centroid of the cross section.

Note the sign convention here. The bending moment M is positive when tensile stress is on the bottom fiber or the center of curvature is above the beam. Positive z is taken to be downward.

On a cross section of a linearly elastic beam having the z axis as a vertical axis of symmetry, the normal stress $\sigma_x = \sigma$ acting on a longitudinal fiber at a distance z from the neutral axis is given by the *flexure formula*,

$$\sigma = Mz/I \tag{3.55}$$

Here M is the net internal bending moment at the section and I is the moment of inertia of the cross section about the neutral axis (y).

The stresses, like the deformations (and strains), vary linearly with the distance from the neutral axis (Fig. 3-28). The stresses are tensile on one side of the neutral axis and compressive (negative) on the other side. The maximum stress for a cross section occurs at the outermost fibers of the beam and is given by

$$\sigma_{\max} = Mc/I \tag{3.56a}$$

or

$$\sigma_{\max} = M/S \tag{3.56b}$$

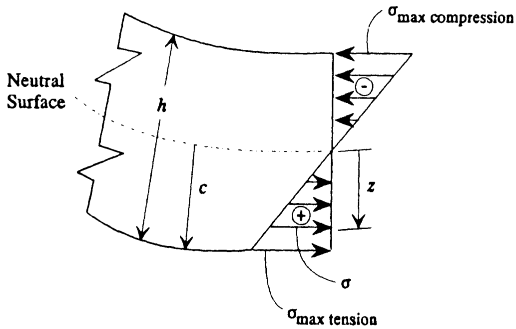


Figure 3-28: Stress distribution on a cross section of a beam.

where c is the distance from the neutral axis to the outermost fiber. The quantity $S = I/c$ is called the *section modulus*, which is a geometric property of the cross section (Chapter 2) and is a measure of the resistance to the development of bending stress.

If a vertical plane is passed through a transversely loaded beam perpendicular to the longitudinal axis, the vertical stresses acting along this plane are called *shear stresses*. Equilibrium requires that the vertical shear stress τ at any point on the cross section is numerically equal to the horizontal shear stress at the same point. These shear stresses, as well as the normal stresses, are assumed to be uniform across the width of the beam. However, the shear stress varies according to the shape of the cross section, as shown in Fig. 3-29.

The shear stress $\tau_{xz} = \tau_{zx} = \tau$ at any point of a prescribed cross section is given by

$$\tau = VQ/Ib \tag{3.57a}$$

where V is the shear force at the section, Q is a first moment (Chapter 2) with respect to the neutral axis of the area beyond the point at which the shear stress is desired,

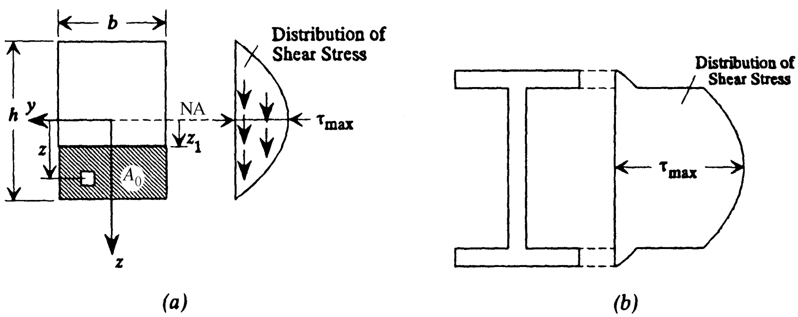


Figure 3-29: Stress distribution on different cross-sectional shapes. A_0 is the shaded area.

I is the moment of inertia about the neutral axis, and b is the width of the section measured at the level at which τ is being determined.

If the shear stress is to be determined at level z_1 of a rectangle cross section, Q must be calculated for the shaded area A_0 of Fig. 3-29a. Equation (2.15a) gives

$$Q = A_0\bar{z}_c = b\left(\tfrac{1}{2}h - z_1\right)\left[z_1 + \tfrac{1}{2}\left(\tfrac{1}{2}h - z_1\right)\right] = \tfrac{1}{2}b\left(\tfrac{1}{4}h^2 - z_1^2\right)$$

From Eq. (3.57a), the desired stress is

$$\tau = \frac{V}{2I}\left(\frac{h^2}{4} - z_1^2\right) \tag{3.57b}$$

and

$$\tau_{\max} = \frac{Vh^2}{8I} = \frac{3}{2}\frac{V}{bh} = \frac{3V}{2A} \tag{3.57c}$$

at the neutral axis ($z_1 = 0$). This equation has been shown to be reasonably accurate for widths equal to or less than the depth ($b \leq h$), but for $b > h$, Eq. (3.57c) should be used with caution. Accurate computational solutions have been developed (Chapter 15).

For a wide-flange I-shaped structural steel, the maximum shear stress given by Eq. (3.57a) is only slightly greater than the average stress obtained by dividing the shear force by the area of the web.

A useful formula in the study of a beam formed of more than one layer (e.g., two boards nailed together), is for the shear flow q. From Eq. (3.57a),

$$q = \tau b = VQ/I \tag{3.57d}$$

This gives the horizontal shear force per unit length of beam that is transmitted between layers of the beam.

Deflection of Simple Beams

The sign convention for forces and displacements of a beam is shown in Fig. 3-30. Applied forces and moments are positive if their vectors are in the direction of a positive coordinate axis. Also, internal shear forces and bending moments acting

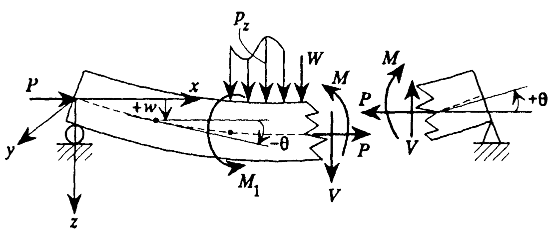


Figure 3-30: Positive applied loadings and internal forces.

on a positive face are positive if their vectors are in positive coordinate directions. The internal forces M, V and applied loads M_1 (concentrated moment, force times length), p_z (loading intensity, force per length), and W (concentrated load, force) shown in Fig. 3-30 are positive.

Positive deflection w is downward (i.e., in the positive coordinate z direction). As shown in Fig. 3-30, θ (radians) is the angle between the axis and the tangent to the curve at a point. Positive and negative θ, which like moments adhere to the right-hand rule, are illustrated.

The basic differential equation relating the deflection w to the internal bending moment M in a beam is

$$\frac{d^2w}{dx^2} = -\frac{M}{EI} \tag{3.58a}$$

where x is the axial coordinate and EI is the flexural rigidity or bending modulus. This relationship applies to a beam that is linearly elastic and where the cross section is symmetric about the xz plane.

For small angles, $\theta \approx \tan\theta = -dw/dx$, that is,

$$\frac{dw}{dx} = -\theta \tag{3.58b}$$

and Eq. (3.58a) appears as $d\theta/dx = M/EI$. The equilibrium equations relate the internal forces M and V and the applied loading density p_z in the form $dV/dx = -p_z$, $dM/dx = V$. If these relations are gathered together,

$$\frac{dw}{dx} = -\theta, \qquad \frac{d\theta}{dx} = \frac{M}{EI}, \qquad \frac{dM}{dx} = V, \qquad \frac{dV}{dx} = -p_z \tag{3.59}$$

These equations are called *governing equations of motion* for the bending of a beam. This first-order form is convenient to handle numerically using a computer. Analytically, it is frequently easier to deal with the higher-order forms:

For Variable EI	For Constant EI
$\theta = -\dfrac{dw}{dx}$	$\theta = -\dfrac{dw}{dx}$
$M = -EI\dfrac{d^2w}{dx^2}$	$M = -EI\dfrac{d^2w}{dx^2}$
$V = \dfrac{dM}{dx} = -\dfrac{d}{dx}\left(EI\dfrac{d^2w}{dx^2}\right)$	$V = -EI\dfrac{d^3w}{dx^3}$
$p_z = -\dfrac{dV}{dx} = \dfrac{d^2}{dx^2}\left(EI\dfrac{d^2w}{dx^2}\right)$	$p_z = EI\dfrac{d^4w}{dx^4}$

(3.60)

These relations are found by successive substitution of Eqs. (3.58) into each other.

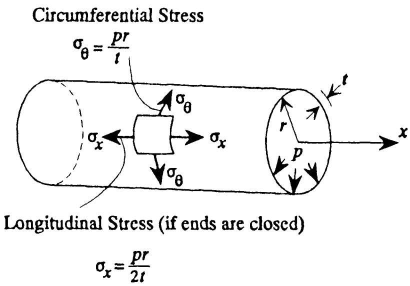


Figure 3-31: Cylinder.

Stress in Pressure Vessels

Thin-walled containers or shells loaded with gas or liquid pressure and having the form of a surface of revolution, such as cylinders and spheres, are discussed in this section.

Cylinder Stress On the wall of a thin-walled cylinder subjected to internal pressure, two stresses in the plane of the wall are of prime interest (Fig. 3-31). These stresses, a longitudinal stress σ_x parallel to the axis of revolution and a hoop or circumferential or cylindrical stress σ_θ perpendicular to σ_x, are called *membrane stresses*. If there are no abrupt changes in wall thickness and the wall is thin (thickness less than about one-tenth the radius r), it can be assumed that the stresses are uniformly distributed through the thickness of the wall and that no other significant stresses occur. Application of the conditions of equilibrium suffices to determine these membrane stresses (Chapter 20). For a cylinder with internal pressure p,

$$\sigma_\theta = pr/t \tag{3.61a}$$

If the ends of the cylinder are closed,

$$\sigma_x = pr/2t \tag{3.61b}$$

The results for the circumferential stress are about 5% in error on the danger side when the thickness is one-tenth the radius of the cylinder ($t = 0.1r$). Shells of greater relative thickness should be analyzed according to bending shell theories (Chapter 20).

Sphere Stress The stresses σ acting in the plane of the sphere wall are the same in all directions under uniform internal pressure (Fig. 3-32):

$$\sigma = pr/2t \tag{3.62}$$

It can be seen that they are one-half the magnitude of the circumferential stresses of the cylinder. When the thickness equals one-fifth the radius of the sphere ($t = 0.2r$), the thin-sphere formula gives values in error by about 2.5% on the danger side. If

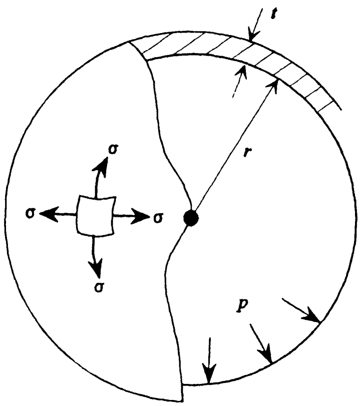


Figure 3-32: Sphere.

the thickness exceeds one-fifth the radius, more accurate formulas should be used (Chapter 20).

Stress for Shells of Revolution A shell of revolution is formed by rotating a plane curve, called the *meridian*, about an axis lying in the plane of the curve (Fig. 3-33). The stresses on an element of a general membrane shell of revolution (Fig. 3-34*a*) are related to the pressure p by

$$\sigma_\phi/R_\phi + \sigma_\theta/R_\theta = p/t \tag{3.63}$$

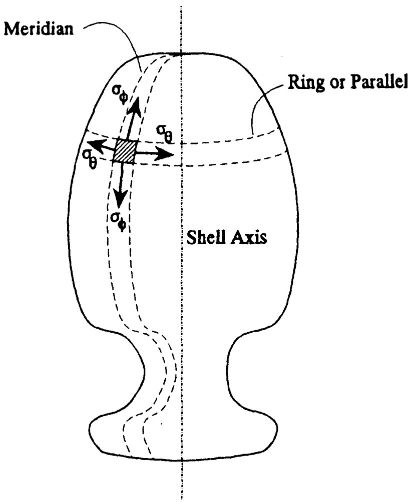


Figure 3-33: Shell of revolution.

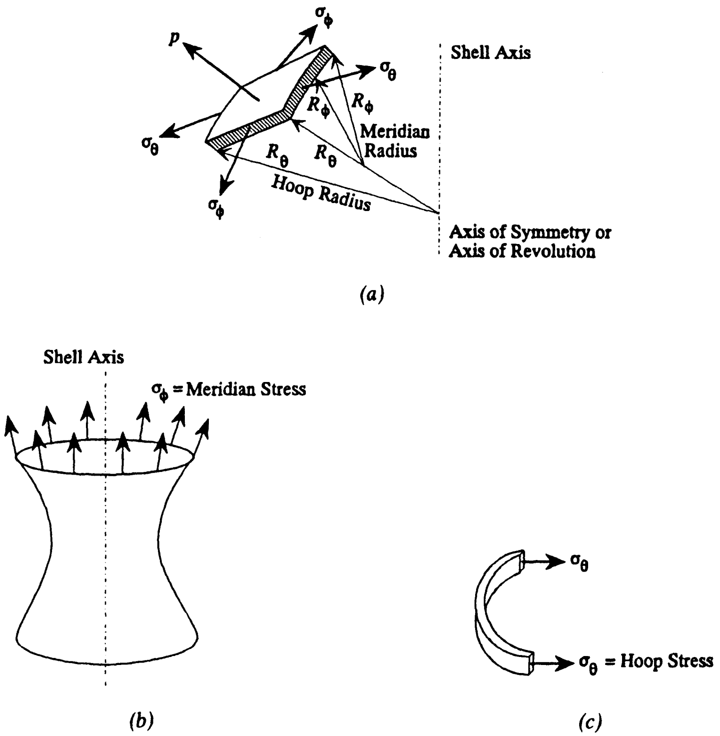


Figure 3-34: Stresses in a shell of revolution: (*a*) shell element; (*b*) meridional stress; (*c*) circumferential stress.

where σ_ϕ is the meridional stress (psi, N/m^2 or Pa) (Fig. 3-34*b*), R_ϕ is the radius of curvature of the meridian, σ_θ is the hoop, ring, or circumferential stress (psi, N/m^2) (Fig. 3-34*c*), and R_θ is the radius of curvature of the section normal to the meridian curve; that is, R_θ is the length of the normal between the surface and the axis of revolution and originates at the shell axis and in general is not perpendicular to the shell axis, whereas the center of curvature for R_ϕ in general will not lie on the shell axis (see Fig. 3-34*a*).

3.13 COMBINED STRESSES

In the most general case, a body may be subjected to a variety of types of loadings, such as a combination of tension, compression, twisting, and bending loads. In such a case, it will be assumed that each load produces the stress that it would if it were the only load acting on the body. As long as linearity prevails, the final stress is then found by careful superposition of the several states of stress.

Frequently, there is little difficulty in identifying the individual states of stress composing a combined stress problem. The appropriate stress formula developed in previous sections should be associated with each load. For example, in a bar subjected simultaneously to tension and torsion loads, the axial normal stress component is $\sigma_x = P/A$, where P is the tensile load and A is the cross-sectional area of the bar. Also present is a shear stress due to the torque, $\tau = Tr/J$, where T is the torque, r the radius of the section, and J the polar moment of inertia. The case above leads to one normal and one shear stress. Normal stresses (e.g., extension and bending stresses), are directly additive, as are shear stresses if they act in the same direction. If not, the methods in Section 3.3 are employed, usually to calculate the principal stresses at a point.

Note that superposition is valid if the material is linearly elastic and if the effect of one type of loading does not influence the internal force corresponding to other loadings of interest.

Example 3.5 Bar under Combined Stresses Find the maximum shear stress on the face of the shaft of circular cross section shown in Fig. 3-35.

At any axial location to the right of the 120 in.-lb torque (Fig. 3-35), we find the internal forces to be $V = 800$ lb, $T = 120$ in.-lb, and $M = 800x$ in.-lb. The shear stresses are given by Eqs. (3.47) and (3.57a). From these formulas the peak torsional stress occurs at the outer fibers, and the shear stress due to V is a maximum at the diameter 1–2 in Fig. 3-35c, where Q is a maximum. The maximum combined shear stress occurs at point 1, where the two peak shear stresses act in the same direction:

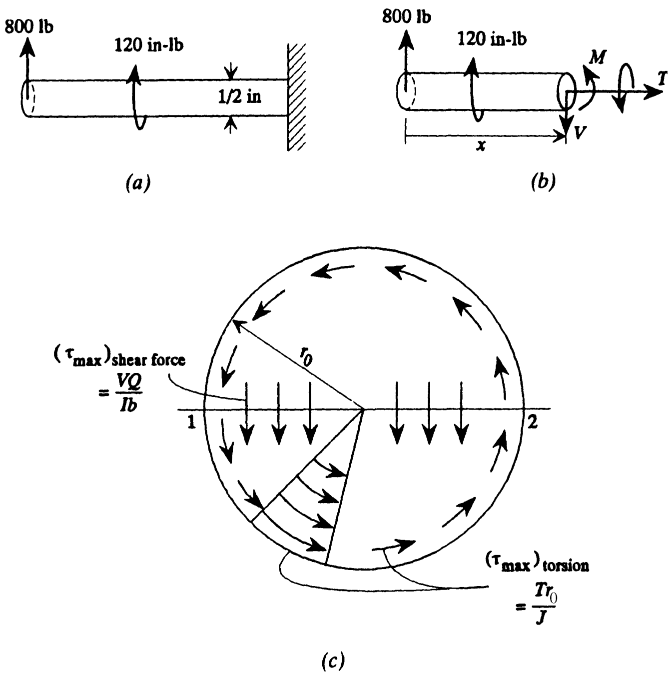


Figure 3-35: Bar under combined stresses.

$$\tau_{\max} = \frac{VQ}{Ib} + \frac{Tr_0}{J} = \frac{V(\pi r_0^2/2)(4r_0/3\pi)}{(\pi r_0^4/4)2r_0} + \frac{Tr_0}{J} = \frac{4V}{3\pi r_0^2} + \frac{Tr_0}{J}$$
$$= \frac{4V}{3A} + \frac{Tr_0}{J} = \frac{4(800)}{3(0.196)} + \frac{120(0.25)}{0.00614} = 10{,}328 \text{ psi} \quad (1)$$

where $A = \frac{1}{4}\pi d^2 = 0.196$ in^2 and $J = \frac{1}{32}\pi d^4 = 0.00614$ in^4. Note the bar also has an axial normal stress due to bending.

Example 3.6 Eccentric Loads A cantilever beam is loaded by a force of 40 kN applied 80 mm from the centroid (Fig. 3-36). Find the maximum normal stress for a vertical cross section. Neglect the weight of the beam.

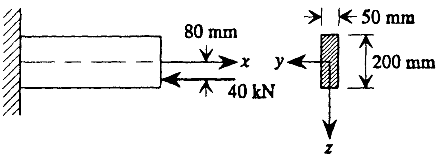


Figure 3-36: Eccentric load.

The eccentric load $P = 40$ kN is statically equivalent to the load P through the centroid and the moment $Pe = 40 \times 80$ mm·kN about a centroid axis. The combined normal stress is

$$\sigma = -\frac{P}{A} - \frac{Pez}{I} = \frac{-40 \text{ kN}}{(200 \text{ mm})(50 \text{ mm})} - \frac{(40 \text{ kN})(80 \text{ mm})z}{\frac{1}{12}(50 \text{ mm})(200 \text{ mm})^3} \quad (1)$$

The peak bending stresses occur at the outer fibers where $z = \pm 100$ mm. Thus, at the bottom fibers,

$$\sigma = -4.0 - 9.6 = -13.6\text{N/mm}^2 = -13.6\text{MN/m}^2 \quad \text{(compression)} \quad (2)$$

At the top fibers,

$$\sigma = -4.0 + 9.6 = 5.6\text{N/mm}^2 = 5.6\text{MN/m}^2 \quad \text{(tension)} \quad (3)$$

Example 3.7 Combined Bending and Torsion of Shafts Show that when a solid circular shaft of diameter d is subjected to a bending moment M and a torque T, (a) the maximum principal stress is equal to $16(M+\sqrt{M^2+T^2})/\pi d^3$ and (b) the maximum shear stress is equal to $16\sqrt{M^2+T^2}/\pi d^3$.

The maximum stresses, which occur at the outer fibers, are given by Eqs. (3.56a) and (3.47) with $J = 2I$ and $r = z = c$:

$$\sigma = Mc/I, \qquad \tau = Tc/J = Tc/2I \tag{1}$$

The maximum principal stress is derived using Eq. (3.13a):

$$\begin{aligned}\sigma_1 &= \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{Mc}{2I} + \sqrt{\left(\frac{Mc}{2I}\right)^2 + \left(\frac{Tc}{2I}\right)^2} \\ &= \frac{16}{\pi d^3}\left(M + \sqrt{M^2 + T^2}\right)\end{aligned} \tag{2}$$

where we have set $c = \frac{1}{2}d$. The peak shear stress is found from Eq. (3.14):

$$\tau_{\max} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \underline{\underline{\frac{16}{\pi d^3}}}\sqrt{M^2 + T^2} \tag{3}$$

For convenient reference, the basic stress formulas considered in this chapter for simple configurations are given in Table 3-3. The basic deformation formulas are given in Table 3-4.

3.14 UNSYMMETRIC BENDING

Normal Stress

The formula for normal stress in straight beams, $\sigma = Mz/I$, is applicable only if the bending moment acts around one of the principal axes of inertia of the cross section. That is, the bending stress theory developed thus far is appropriate for a symmetric cross section bent in its plane of symmetry.

Consider the more general case of an unsymmetric cross section with positive (tensile) axial P and bending moment components $M_y = M$ and M_z. The formula

$$\sigma = \frac{P}{A} + \frac{M_y I_z + M_z I_{yz}}{I_z I_y - I_{yz}^2} z - \frac{M_z I_y + M_y I_{yz}}{I_z I_y - I_{yz}^2} y \tag{3.64}$$

applies. The coordinates y, z are measured from axes passing through the centroid of the cross section. The moments of inertia $I_y = I$, I_z, I_{yz} are taken about these axes.

Loading in One Plane If the bending moment M_z is zero, Eq. (3.64) reduces to a formula applicable to an unsymmetric section loaded in a single plane:

$$\sigma = P/A + M_y(I_z z - I_{yz} y)/(I_z I_y - I_{yz}^2) \tag{3.65}$$

Principal Axes Suppose that y, z correspond to principal axes of inertia through the centroid. Then $I_{yz} = 0$ and Eq. (3.64) becomes

$$\sigma = P/A + M_y z/I_y - M_z y/I_z \tag{3.66}$$

where the bending moments have been resolved into components along the principal axes.

Bending about a Single Axis Equation (3.64) reduces to the usual bending stress formula of Eq. (3.55) if the bending moment acts around a single principal axis of inertia through the centroid. We use Eq. (3.66) with

$$M_y = M, \qquad P = 0, \qquad M_z = 0, \qquad I_y = I$$

Then

$$\sigma = Mz/I$$

Example 3.8 Unsymmetric Bending Consider the beam section in Fig. 3-37*a*. From the formulas of Chapter 2,

$$I_z = \tfrac{1}{12}th^3, \qquad I_y = \tfrac{1}{3}th^3, \qquad I_{yz} = \tfrac{1}{8}th^3 \tag{1}$$

To compute the bending stresses, use Eq. (3.65) with $P = 0$, $M_y = M$,

$$\sigma = \frac{M_y(I_z z - I_{yz} y)}{I_z I_y - I_{yz}^2} = \frac{M}{th^3}\left(\frac{48}{7}z - \frac{72}{7}y\right) \tag{2}$$

which is plotted in Fig. 3-37*b*. The peak stresses occur at extreme fibers. At point A, with $z = y = \frac{1}{2}h$, we find $\sigma_A = -12M/7th^2$. At B, with $z = \frac{1}{2}h$ and $y = 0$, $\sigma_B = 24M/7th^2$ (see Fig. 3-37*b*).

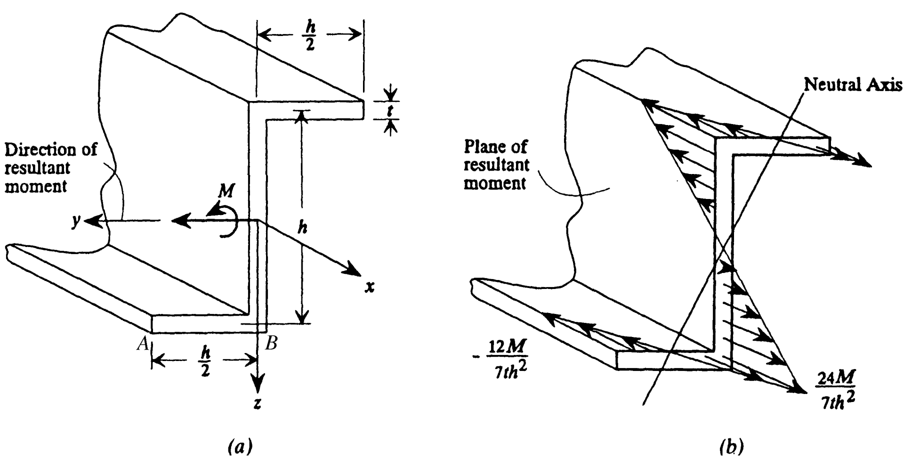


Figure 3-37: Example 3.8: unsymmetric bending.

If formula 4 in Table 3-3, which does not take the lack of symmetry into account, had been used, then

$$\sigma_A = \sigma_B = \left(\frac{Mz}{I_y}\right)_{z=h/2} = \frac{\frac{1}{2}Mh}{\frac{1}{3}th^3} = \frac{3M}{2th^2} \tag{3}$$

Comparison of this with the correct values of σ_A and σ_B shows that errors of 188 and 56%, respectively, would occur.

Shear Stress

The familiar formula for shear stress in straight beams, $\tau = VQ/Ib$, applies to symmetric sections in which the shear force V is along one of the principal axes of inertia of the cross section. For an unsymmetric cross section with positive shear forces V_z and V_y, the average shear stress is given by

$$\tau = \frac{I_z Q_y - I_{yz} Q_z}{b(I_z I_y - I_{yz}^2)} V_z + \frac{I_y Q_z - I_{yz} Q_y}{b(I_z I_y - I_{yz}^2)} V_y \tag{3.67}$$

where Q_y and Q_z are first moments of inertia of the area beyond the point at which τ is calculated (Fig. 3-38). These first moments are defined by Eq. (2.15).

The coordinates y, z in Eq. (3.67) are referred to axes passing through the centroid of the cross section. If the width b is chosen parallel to the y axis, Eq. (3.67) gives the stress τ_{zx}. If b is parallel to the z axis, Eq. (3.67) corresponds to τ_{xy}. Moreover, b can be chosen such that Eq. (3.67) gives the average shear stress in any direction. This is accomplished by selecting b to be the section width at the point where the stress is sought. This width is taken in a direction perpendicular to the desired stress. If the shear stress along the line 1–2 of the section in Fig. 3-38 is to be computed, b

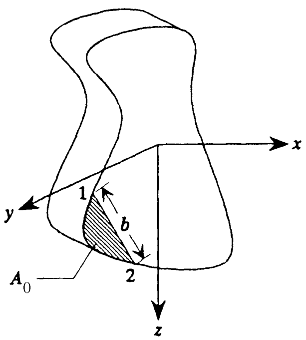


Figure 3-38: Shear stress.

should be selected as indicated. This fixes area A_0 and also establishes Q_y and Q_z. Note that according to this formula, the average shear stress is constant along 1–2. Hence, only when the actual shear stress is constant along 1–2 is the average shear stress of Eq. (3.67) equal to the actual shear stress on b. Equation (3.67) is normally considered to be reasonably accurate for thin-walled sections and somewhat less accurate for thick sections. More accurate stresses are provided by the computer program discussed in Chapter 15.

Equation (3.67) is usually employed to calculate the shear stress or shear flow in thin-walled open sections. This relationship reduces to $\tau = VQ/Ib$ if the loading is in the xz plane ($V_y = 0$) and z and y are the principal axes of inertia ($I_{yz} = 0$).

Example 3.9 Shear Stress in Unsymmetric Bending Find the shear flow in the beam section in Fig. 3-39a due to the shear force V_z.

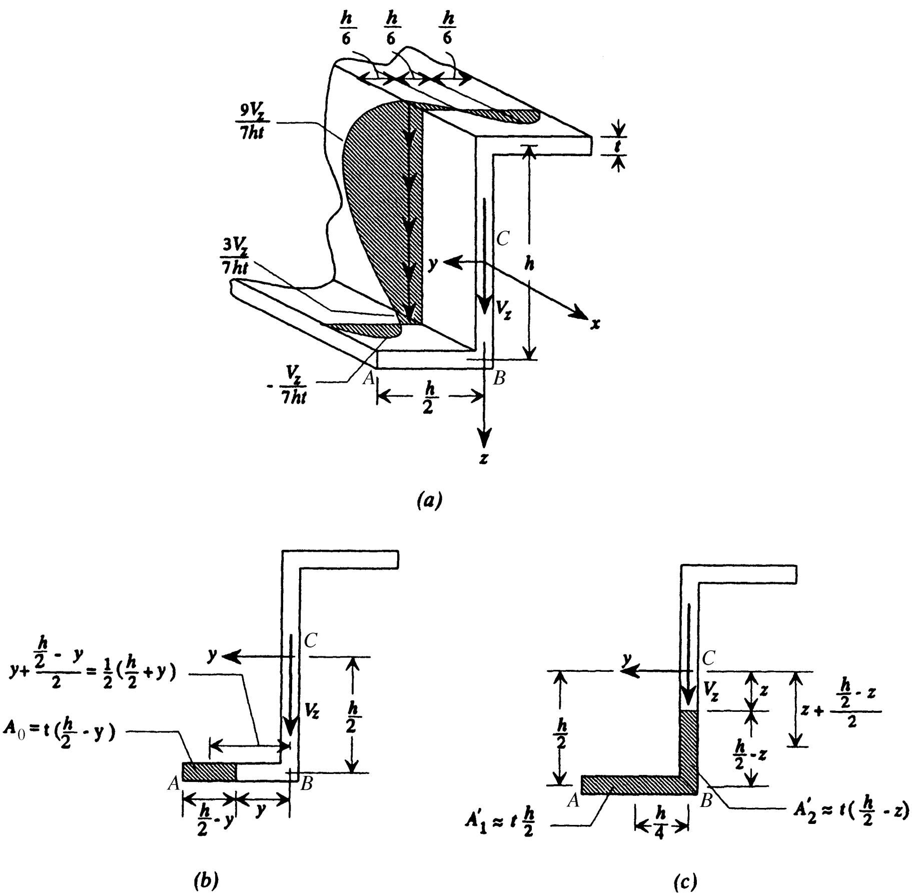


Figure 3-39: Example 3.9.

The shear flow is calculated from Eq. (3.67) using $q = \tau b$. The moments of inertia are given by Eq. (1) of Example 3.8. Equation (3.67), with $b = t$, reduces to

$$\tau = \frac{48Q_y - 72Q_z}{7t^2h^3}V_z \tag{1}$$

The first moments Q_y and Q_z of Eq. (2.15) are taken about y, z coordinates passing through the centroid. For a point in the flange between A and B (Fig. 3-39*b*),

$$\begin{aligned} Q_z &= \int_{A_0} y\,dA = \frac{1}{2}\left(\frac{1}{2}h + y\right)A_0 \\ &= \frac{1}{2}\left(\frac{1}{2}h + y\right)t\left(\frac{1}{2}h - y\right) = \frac{1}{2}t\left[\left(\frac{1}{2}h\right)^2 - y^2\right] \\ Q_y &= \int_{A_0} z\,dA = \frac{1}{2}hA_0 = \frac{1}{2}ht\left(\frac{1}{2}h - y\right) \end{aligned} \tag{2}$$

From (1), the stress between A and B is given by

$$\tau = \frac{12}{7th^3}\left(\frac{h^2}{4} - 2hy + 3y^2\right)V_z \tag{3}$$

which is a parabola in y.

For a point in the web between C and B in Fig. 3-39*c*,

$$\begin{aligned} Q_z &= \int_{A_0} y\,dA = \frac{h}{4}A'_1 + (0)A'_2 = \frac{h}{4}\left(t\frac{h}{2}\right) = t\frac{h^2}{8} \\ Q_y &= \int_{A_0} z\,dA = \frac{h}{2}A'_1 + \left(z + \frac{h/2 - z}{2}\right)A'_2 = \frac{t}{2}\left(\frac{3h^2}{4} - z^2\right) \\ \tau &= \frac{24}{7th^3}\left(\frac{3}{8}h^2 - z^2\right)V_z \end{aligned} \tag{4}$$

where A'_1 is the area of the lower flange and A'_2 is the area of that portion of the web beyond the point at which τ is calculated.

The distribution of shear stress is shown in Fig. 3-39*a*. The peak value of $9V_z/7ht$ occurs at C, the centroid. If formula 5 in Table 3-3 were used to calculate the stress, the maximum value would occur at C. Using (4) above,

$$\tau = \frac{VQ}{Ib} = \frac{V_zQ_y}{I_yt} = \frac{\frac{1}{2}V_zt(\frac{3}{4}h^2 - z^2)}{(\frac{1}{3}th^3)t} = \frac{3}{2th^3}\left(\frac{3}{4}h^2 - z^2\right)V_z \tag{5}$$

and at $z = 0$, $\tau_{\max} = 9V_z/8ht$. This is 12.5% in error relative to the more exact value found using Eq. (3.67).

3.15 THEORIES OF FAILURE

Concept of Failure

Structural members and machine parts may fail to perform their intended functions if excessive elastic deformation, yielding (plastic deformation), or fracture (break) occurs. For a failure-safe design, the engineer must determine possible modes of failure of the structural or machine system and then establish suitable failure criteria that accurately predict the various modes of failure. The determination of modes of failure [3.8] requires extensive knowledge of the response of material or a structural system to loads. In particular, it may require a comprehensive stress analysis of the system. The mode of failure depends on the type of material used and the manner of loading (e.g., static, dynamic, and fatigue).

Two types of excessive elastic deformation result in structural failure:

1. Deformation satisfying the usual conditions of equilibrium, such as deflection of a beam or angle of twist of a shaft under gradually applied (static) loads. The ability to resist such deformation is referred to as the stiffness of a member. Furthermore, there can be excessive deformations associated with the amplitudes of the vibration of a machine member.
2. Buckling or an inordinately large displacement under conditions of unstable equilibrium that may occur in a slender column when the axial load exceeds the Euler critical load, in a thin plate when the in-plane forces exceed the critical load, or when the external pressure on a thin-walled shell exceeds a critical value. This is a form of instability, referred to as *bifurcation*.

To ascertain if it will serve its purpose, a load-carrying solid must be investigated from the standpoint of strength in addition to the possibility of the stiffness and stability failures considered above. A discussion of strength-related failure follows.

Yielding failure is due to plastic deformation of a significant part of a member, sometimes called *extensive yielding* to distinguish it from (localized) yielding of a small part of a member. Yielding under room and elevated temperatures is discussed in Chapter 4. Yielding occurs when the elastic limit (or yielding point) of the material has been exceeded. As indicated in Chapter 4, in a ductile metal under conditions of static loading at room temperature, yielding rarely results in fracture because of the strain-hardening effect. For simple tensile loading, failure by excessive plastic deformation is controlled by the yield strength of the metal. However, for more complex loading conditions, the yield strength must be used with a suitable criterion, a "theory of failure," which is discussed later in this section. At temperatures significantly greater than room temperature, metals no longer exhibit significant hardening. Instead, metals can deform continuously at constant stress levels in a time-dependent yielding known as *creep*.

Members can fracture before failure defined by excessive elastic deformation or yielding can occur. The mechanisms of this fracture include the following:

1. Rapid fracture of brittle materials
2. Fatigue of progressive fracture
3. Fracture of flawed members
4. Creep at elevated temperatures

Fatigue deserves special attention because the magnitude of the repetitive load need not be high enough to cause static fracture (i.e., the stress may be relatively low). But under lengthy vibratory loading, fatigue cracks can form. Fatigue fracture is often ranked as the most serious type of fracture in machine design simply because it can occur under normal operating conditions. Fracture and fatigue are discussed in Chapter 7. Creep is discussed in Chapter 4. Failure theories for yield are treated in the following subsections. By replacing the yield stress by another critical stress level (e.g., the ultimate stress), these theories are often considered to be applicable to failures other than yield.

Tensile tests provide the most commonly available information about the failure level of a material. The problem arises when an attempt is made to relate these tensile data to a combined stress situation. In some combined stress cases tests can be performed to determine the yield stress. Usually, it is not convenient, or even possible, to conduct a suitable model test; consequently, it is necessary to develop a relationship between stress under complicated stress conditions and the behavior of a material in simple tension or compression.

For the theories considered here, it is assumed that the tension or compression critical stresses σ_{ys} (yield stress) or σ_u (ultimate stress) are available as properties found from simple material tests. In developing the various failure criteria, it is convenient to use the fact that any state of stress at a point can be reduced through a rotation of coordinates to a state of stress involving only the principal stresses σ_1, σ_2, and σ_3. Often, these principal stresses are output by general-purpose structural analysis programs. The same reasoning applies to strains.

Maximum-Stress Theory In the maximum-stress theory, or Rankine theory, the maximum principal stress is taken as the criterion of failure. For the moment, failure is to be defined in terms of yielding, although the same theory applies if the yield stress is replaced by another stress level, such as the ultimate stress. For the maximum-stress theory, yield occurs at a point in the structure when one of the principal stresses at this point, which is subjected to combined stresses, reaches the yield strength in simple tension (σ_{ys}) or compression for the material. According to this theory, yielding is not affected by the level of the other principal stresses. Thus, for material whose tension and compression properties are the same, the failure criterion

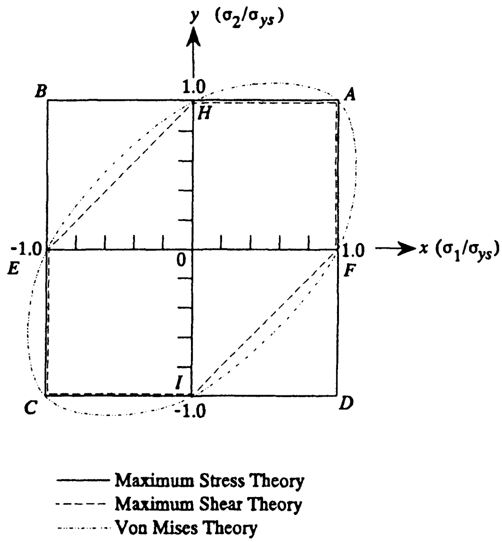


Figure 3-40: Graphical representation of theories of failure in a two-dimensional state of stress.

is defined as

$$\sigma_1 = \sigma_{ys} \quad \text{or} \quad |\sigma_3| = \sigma'_{ys} \tag{3.68}$$

where σ_{ys} and σ'_{ys} are the yield stresses in simple tension and compression, respectively. The principal stresses are so arranged that their algebraic values satisfy the relation $\sigma_1 > \sigma_2 > \sigma_3$.

Maximum-stress theory can readily be illustrated. For example, a graphical representation in a two-dimensional state of stress is shown in Fig. 3-40. The locus of failure points is the square $ABCD$.

Maximum-Strain Theory The maximum-strain theory, considered to be due to Saint-Venant, postulates that a ductile material begins to yield when the maximum extensional strain at a point reaches the yield strain in simple tension, or when the minimum strain (shortening) equals the yield point strain in simple compression. By means of Hooke's laws, for $\sigma_1 > \sigma_2 > \sigma_3$, this failure criterion is embodied in the equations

$$\sigma_1 - \nu(\sigma_2 + \sigma_3) = \sigma_{ys}, \qquad |\sigma_3 - \nu(\sigma_1 + \sigma_2)| = \sigma'_{ys} \tag{3.69}$$

The maximum-strain theory is not considered to be reliable in many instances.

Maximum-Shear Theory The maximum-shear theory, or Tresca or Guest's theory, assumes that failure occurs in a body subjected to combined stresses when the maximum shear stress at a point [e.g., $\frac{1}{2}(\sigma_1 - \sigma_2)$], reaches the value of shear failure stress of the material in a simple tension test [e.g., $\frac{1}{2}\sigma_{ys}$]. Therefore, failure under combined stresses is decided by the condition

$$\sigma_{\max} - \sigma_{\min} = \sigma_{ys} \tag{3.70a}$$

where $\sigma_{\max}$ and $\sigma_{\min}$ are the maximum and minimum principal stresses, respectively.

The term $\sigma_{\max} - \sigma_{\min}$ can also be expressed as

$$\max(|\sigma_1 - \sigma_2|,\ |\sigma_2 - \sigma_3|,\ |\sigma_3 - \sigma_1|) \tag{3.70b}$$

The largest of these absolute values is sometimes referred to as the *stress intensity*. This quantity is often computed by general-purpose analysis software.

It is important to note that for the case $\sigma_1 > \sigma_2 > \sigma_3$, the failure criterion would be

$$\sigma_1 - \sigma_3 = \sigma_{ys} \tag{3.70c}$$

A plot of this theory for a two-dimensional state of stress is given in Fig. 3-40. The locus of failure points is the polygon *AHECIFA*.

von Mises Theory The *von Mises theory*, also called the *Maxwell–Huber–Hencky–von Mises theory*, *octahedral shear stress theory*, and *maximum distortion energy theory*, states that failure at a particular location occurs when the energy of distortion reaches the same energy for failure in tension. That is, failure takes place when the principal stresses are such that

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 = 2\sigma_{ys}^2 \tag{3.71a}$$

This relation holds regardless of the relative magnitude of σ_1, σ_2, and σ_3.

The quantity

$$\left\{\tfrac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2\right]\right\}^{1/2} = \sigma_e \tag{3.71b}$$

is often referred to as the *equivalent stress*. This is sometimes available as output of general-purpose structural analysis software.

In a two-dimensional state of stress ($\sigma_3 = 0$), Eq. (3.71a) becomes

$$\sigma_{ys}^2 = \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 \tag{3.72}$$

This relationship is plotted in Fig. 3-40.

Mohr's Theory Mohr's theory, also called *Coulomb–Mohr theory* or *internal-friction theory*, is based on the results of the standard tension and compression tests, which give the tensile and compressive strengths σ_{ys} and σ'_{ys}. Two Mohr circles for these experiments can be plotted on the same diagram. A pair of lines AB and

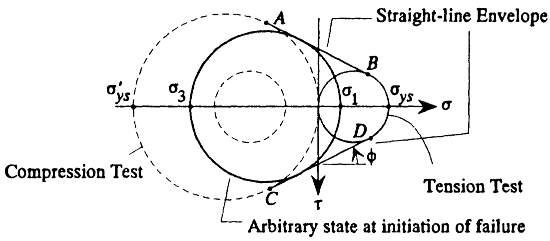


Figure 3-41: Mohr's theory of failure.

CD (Fig. 3-41) are drawn tangent to the two Mohr circles. Mohr's theory states that failure of an isotropic material, either by fracture or by the onset of yielding, will occur at a point where the largest Mohr circle for this point (having diameter $\sigma_1 - \sigma_3$, as in Fig. 3-41) touches a failure envelope. Any "interior" circle, such as the dashed one in Fig. 3-41, represents a state of stress that is safe, while the solid circle represents a state of stress that is in failure. It can be shown that failure occurs when

$$\sigma_1/\sigma_{ys} + \sigma_3/\sigma'_{ys} \geq 1 \tag{3.73}$$

where $\sigma_{ys} > 0$ and $\sigma'_{ys} < 0$ and the maximum and minimum principal stresses σ_1 and σ_3 carry their algebraic signs. In plane stress problems, if all normal stresses are tensile, Eq. (3.73) coincides with the maximum-stress theory ($\sigma_1 \geq \sigma_{ys}$). For ductile materials, it is usually assumed that $\sigma_{ys} = -\sigma'_{ys}$, so that Eq. (3.73) becomes $\sigma_1 - \sigma_3 \geq \sigma_{ys}$.

Validity of Theories

The appropriate failure theory to be used in a given design situation depends on the mode of failure. A theory that works for ductile failure may not be appropriate for brittle failure. A single theory may not always apply to a given material because the material may behave in a ductile fashion under some conditions and in a brittle fashion under others (see Chapter 4). For the foregoing theories, the material is assumed to be isotropic. These theories of failure pertain to material failure rather than to structural failure by such modes as buckling or excessive elastic deformation.

A comparison has been made [3.2] of experimental yield stresses for several metals under biaxial stress conditions with some of the failure theories described above. The results, which are for room temperature and slow loading, seem to indicate somewhat better agreement with von Mises theory than with maximum-shear theory.

Maximum-stress and maximum-strain theory are often applicable to brittle failure of materials, so that σ_u often replaces σ_{ys} in Eqs. (3.68) and (3.69). Maximum-strain theory has been shown not to be reliable in many instances. Maximum-shear theory is applied frequently in machine design for ductile materials where $\sigma_{ys} = \sigma'_{ys}$.

Maximum-shear theory has the advantage over von Mises theory that the stresses appear in a linear fashion.

Mohr's theory is generally used for brittle materials, which are much stronger in compression than in tension (e.g., for cast iron).

3.16 APPLICATION OF FAILURE THEORIES

The following examples illustrate use of the failure theories discussed above.

Example 3.10 Internal Pressure of a Cylindrical Vessel A cylindrical pressure vessel 80 in. in diameter and 1 in. thick is made of steel with a yield stress in tension of 35,000 psi. Determine the internal pressure that will produce yielding by using the von Mises theory of failure as the yield criterion.

From the stress formulas for thin-walled pressure vessels presented previously, the principal stresses at any point in a cylinder (Fig. 3-31) will be the circumferential stress σ_θ, the longitudinal stress σ_x, and the radial stress σ_r. Let $\sigma_1 = \sigma_\theta$, $\sigma_2 = \sigma_x$, and $\sigma_3 = \sigma_r$. Equations (3.61) give

$$\sigma_1 = pr/t, \qquad \sigma_2 = pr/2t \tag{1}$$

The stress σ_r is small $(0 \le \sigma_r \le p)$ relative to σ_θ and σ_x and is neglected; that is,

$$\sigma_3 = 0 \tag{2}$$

Substituting (1) and (2) in Eq. (3.71a) gives

$$\begin{aligned}(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 &= 2(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2) \\ &= 2\left[\left(\frac{pr}{t}\right)^2 - \frac{pr}{t}\frac{pr}{2t} + \left(\frac{pr}{2t}\right)^2\right] = 2\sigma_{ys}^2\end{aligned}$$

Therefore,

$$p = \sqrt{\frac{4}{3}\frac{t^2}{r^2}(\sigma_{ys}^2)} = \frac{2}{\sqrt{3}}\frac{t\sigma_{ys}}{r} = \frac{2}{\sqrt{3}}\frac{(1)(35{,}000)}{80/2} \approx 1010 \text{ psi} \tag{3}$$

According to von Mises theory, this p gives the pressure value that would initiate yielding of the cylinder.

If the maximum-stress theory of failure [Eq. (3.68)] and the maximum-shear theory [Eq. (3.70c)] are used, the internal pressure that will produce yielding in both cases is

$$p = \frac{t\sigma_{ys}}{r} = \frac{1(35{,}000)}{80/2} = 875 \text{ psi} \tag{4}$$

In establishing this relationship, remember that $\sigma_3 = 0$.

The results in (3) and (4) indicate that in this case, using the maximum stress theory and maximum shear theory is more conservative than using the von Mises theory.

Example 3.11 Application of Tresca and von Mises Theories If the yield strength of a material in a tensile test is $\sigma_{ys} = 140\text{MN/m}^2$, determine the largest safe shear stress τ in a cylinder of the same material in torsion.

In the simple tension test, the state of stress is $\sigma_1 = \sigma_{ys} = 140\text{MN/m}^2$, $\sigma_2 = \sigma_3 = 0$. From Eq. (3.70a),

$$\sigma_{\text{max}} - \sigma_{\text{min}} = \sigma_1 - 0 = \sigma_{\text{ys}} \tag{1}$$

For pure torsion of the cylinder $\tau = Tr/J$ (Table 3-3). The principal stresses are, by Eqs. (3.13),

$$\sigma_{\text{max}} = \sigma_1 = \tau, \qquad \sigma_{\text{min}} = \sigma_2 = -\tau, \qquad \sigma_3 = 0 \tag{2}$$

Therefore,

$$\sigma_{\text{max}} - \sigma_{\text{min}} = 2\tau \tag{3}$$

Use of the Tresca theory yields [see Eq. (3.70a)], from (1) and (3),

$$2\tau = \sigma_{ys} \quad \text{or} \quad \tau = \tfrac{1}{2}\sigma_{ys} = \tfrac{140}{2} = 70\text{MN/m}^2 \tag{4}$$

If the von Mises theory is to be used, the equivalent stress σ_e in Eq. (3.71b) is evaluated for the two states of stress. For simple tension

$$\sigma_e = \frac{1}{\sqrt{2}}(\sigma_{ys}^2 + \sigma_{ys}^2)^{1/2} = \sigma_{ys} \tag{5}$$

For torsion of the cylinder, from (2),

$$\sigma_e = \frac{1}{\sqrt{2}}(4\tau^2 + \tau^2 + \tau^2)^{1/2} = \sqrt{3}\,\tau \tag{6}$$

By the von Mises theory, equating (5) and (6), we have

$$\sqrt{3}\,\tau = \sigma_{ys} \quad \text{or} \quad \tau = 0.577\sigma_{ys} = 80.83\text{MN/m}^2 \tag{7}$$

Of course, the same result is obtained by applying Eq. (3.71a) directly.

The results in (4) and (7) indicate that for torsion of a cylinder, Tresca (maximum-shear) theory is more conservative than von Mises theory.

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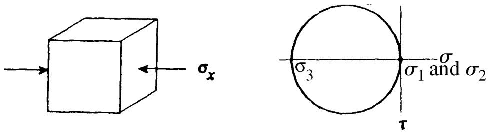
3

Tables

TABLE 3-1 MOHR'S CIRCLES FOR SOME COMMON STATES OF STRESS

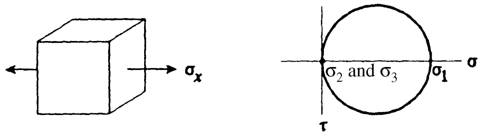
1.
Uniaxial compression

$\sigma_x = \sigma_3$,
$\sigma_1 = \sigma_2 = 0$, so that
$\tau_{max} = (\sigma_{max} - \sigma_{min})/2 = \sigma_3/2$



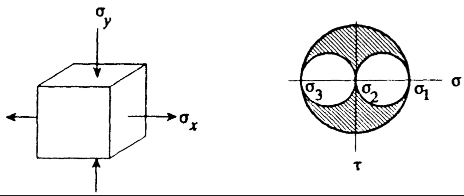
2.
Uniaxial tension

$\sigma_x = \sigma_1$,
$\sigma_2 = \sigma_3 = 0$, giving
$\tau_{max} = (\sigma_{max} - \sigma_{min})/2 = \sigma_1/2$



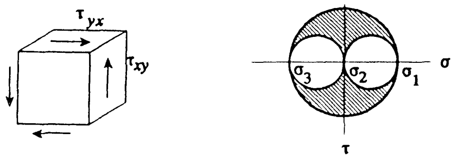
3.
Pure shear

$\sigma_x = -\sigma_y = \sigma_1 = -\sigma_3$,
$\sigma_2 = 0$, so that
$\tau_{max} = \sigma_1$



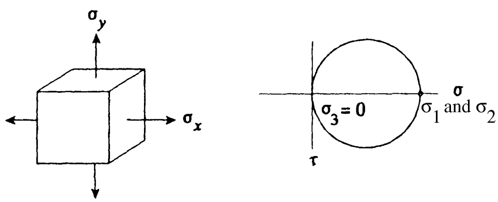
4.
Pure shear

$\tau_{xy} = \tau_{yx}$,
$\sigma_2 = 0$, so that
$\tau_{max} = \sigma_1$



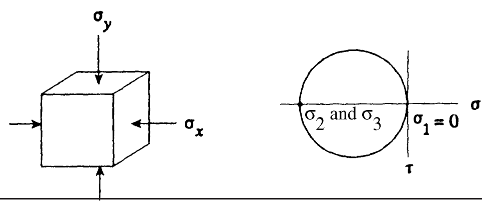
5.
Equal biaxial tension

$\sigma_x = \sigma_y = \sigma_1 = \sigma_2$, giving
$\tau_{max} = \sigma_1/2$

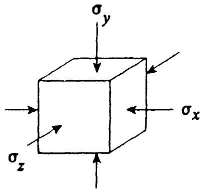


6.
Equal biaxial compression

$\sigma_x = \sigma_y = \sigma_2 = \sigma_3$, so that
$\tau_{max} = \sigma_2/2$



7.
Equal triaxial compression
$\sigma_x = \sigma_y = \sigma_z = \sigma_1 = \sigma_2 = \sigma_3$



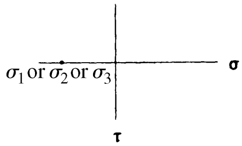


TABLE 3-2 RELATIONSHIPS BETWEEN COMMONLY USED MATERIAL CONSTANTS[a]

1. Shear modulus $G\ (F/L^2)$ $$G = \frac{E}{2(1+\nu)}$$

2. Lamé coefficient $\lambda\ (F/L^2)$ $$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

3. Bulk modulus $K\ (F/L^2)$ $$K = \frac{E}{3(1-2\nu)}$$

where E = modulus of elasticity (F/L^2)
ν = Poisson's ratio

[a]The units are given in parentheses, using L for length and F for force.

TABLE 3-3 BASIC STRESS FORMULAS

Bars of Linearly Elastic Material

1. Extension: $\sigma = P/A$
2. Torsion: $\tau = Tr/J$ (circular section)
3. Torsion: $\tau = T/2A^*t$ (closed, thin-walled section)
4. Bending: $\sigma = Mz/I$
5. Shear: $\tau = VQ/Ib$

where
σ = normal axial stress = σ_x
τ = shear stress
P = axial force
T = axial torque
V = vertical shear force = V_z
M = bending moment in vertical plane = M_y
A = cross-sectional area
A^* = enclosed area
z = vertical coordinate from neutral axis
I = moment of inertia about neutral axis
J = torsional constant = polar moment of inertia for circular cross section
b = width of cross section
r = radius
Q = first moment with respect to neutral axis of area beyond point at which τ is calculated
t = wall thickness

Shells

6. Cylinder: $\sigma_\theta = pr/t$, $\sigma_x = pr/2t$
7. Sphere: $\sigma = pr/2t$

where
σ_θ = hoop stress in cylinder wall
σ_x = longitudinal stress in cylinder wall
σ = membrane stress in sphere wall
p = internal pressure
t = wall thickness
r = radius

TABLE 3-4 BASIC DEFORMATION FORMULAS: BARS OF LINEARLY ELASTIC MATERIAL[a]

1. Extension: $\Delta = PL/AE$

2. Torsion: $\phi = TL/GJ$

3. Bending: $\dfrac{d^4w}{dx^4} = \dfrac{p_z}{EI} \qquad \theta = -\dfrac{dw}{dx} \qquad M = -EI\dfrac{d^2w}{dx^2} \qquad V = -EI\dfrac{d^3w}{dx^3}$

where A = original cross-sectional area (L^2)
Δ = elongation (L)
E = modulus of elasticity (F/L^2)
ϕ = angle of twist
G = shear modulus (F/L^2)
J = torsional constant (L^4)
L = original length (L)
I = moment of inertia about neutral axis (L^4)
P = axial force (F)
p_z = applied loading density (F/L)
T = torque (LF)
w = deflection (L)

[a] The units are given in parentheses using L for length and F for force.