M.Sc. II Sem. (Mathematics)

Paper 1st - Advanced Abstract Algebra-II

Unit - V

Reference Book :

- **I.N. Herstein, Topics in Algebra, Wiley Easter Ltd., New Delhi, 1975.**
- **Vivek Sahai and Vikas Bist, *Algebra*, Narosa Publishing House, New Delhi, 1999.**

Notation. The matrix of T under given basis of V is denoted by m(T).

We know that for determining a transformation $T \in A(V)$ it is sufficient to find out the image of every basis element of V. Let $v_1, v_2, \ldots, v_n$ be the basis of V over F and let

$$v_1T = \alpha_{11}v_1 + \alpha_{12}v_2 + \ldots + \alpha_{1n}v_n$$

$$\ldots \ldots \ldots \ldots \ldots$$

$$v_iT = \alpha_{i1}v_1 + \alpha_{i2}v_2 + \ldots + \alpha_{in}v_n$$

$$\ldots \ldots \ldots \ldots \ldots$$

$$v_nT = \alpha_{n1}v_1 + \alpha_{n2}v_2 + \ldots + \alpha_{nn}v_n$$

Then matrix of T under this basis is

$$m(t) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{i1} & \alpha_{i2} & \ldots & \alpha_{in} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{n1} & \alpha_{n2} & \ldots & \alpha_{nn} \end{pmatrix}_{n \times n} .$$

Definition. Let T be a linear transformation on V over F. The matrix of T in the basis $v_1, v_2, \ldots, v_n$ is called triangular if

$$v_1T = \alpha_{11}v_1$$

$$v_2T = \alpha_{21}v_1 + \alpha_{22}v_2$$

$$\ldots \ldots \ldots \ldots$$

$$v_iT = \alpha_{i1}v_1 + \alpha_{i2}v_2 + \ldots + \alpha_{ii}v_i$$

$$\ldots \ldots \ldots \ldots$$

$$v_nT = \alpha_{n1}v_1 + \alpha_{n2}v_2 + \ldots + \alpha_{nn}v_n.$$

Definiton. A transformation $T \in A(V)$ is called **nilpotent** if $T^n = 0$ for some positive integer n.

Definition. If $T \in A(V)$ is nilpotent then k is called **index of nilpotence** of T if $T^k = 0$ but $T^{k-1} \neq 0$.

Theorem. Prove that all the characteristic roots of a nilpotent transformation $T \in A(V)$ lies in F.

Proof. Since T is nilpotent, let r be the index of nilpotence of T. Then $T^r = 0$.

Let λ be the characteristic root of T, then there exist $v(\neq 0)$ in V such that

$$vT = \lambda v.$$

As $vT^2 = (vT)T = (\lambda v)T = \lambda(vT) = \lambda\lambda v = \lambda^2 v$.

Therefore, continuing in this way, we get

$$vT^3 = \lambda^3 v, \ldots, \; vT^r = \lambda^r v .$$

Since $T^r = 0$, hence $vT^r = v0 = 0$ and hence

$\lambda^r v = 0$. But $v \neq 0$, therefore, $\lambda^r = 0$ and hence $\lambda = 0$, which all lies in F.

Definition. Let $T \in A(V)$ is nilpotent transformation with index of nilpotence n_1. Then there exist subspace $V_1, V_2, \ldots, V_k$ of dimensions $n_1, n_2, \ldots, n_k$ respectively, each V_i is invariant under T such that $V = V_1 \oplus V_2 \oplus \ldots \oplus V_k$, $n_1 \geq n_2 \geq \ldots \geq n_k$ and $\dim V = n_1 + n_2 + \ldots + n_k$. These integers $n_1, n_2, \ldots, n_k$ are called **invariants of T**.

Remarks.

- Invariants of a nilpotent transformation are unique.
- Transformations S and $T \in A(V)$ are similar if and only if they have same invariants.

Definition. Let G be a finite abelian group and p be a prime number dividing $|G|$. We define

$$G_p = \{x \in G \mid p^r x = 0 \text{ for some } r \in Z\}.$$

Clearly, $0 \in G_p$. If $x, y \in G_p$, then $p^r x = 0$ and $p^s y = 0$, for some $r, s \in Z$, and so

$$p^{r+s}(x - y) = p^s(p^r x) - p^r(p^s x) = 0.$$

Thus, G_p is a subgroup of G.

It may be noted that G_p is a Sylow p-subgroup of G.

In the context of abelian group, we call G_p a **p-primary component** of G.

Primary Decomposition Theorem.

If G is a finite abelian group, then G is a direct sum of its p-primary components.