

Differential Equations of second order:
(with constant coefficients)

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = Q$$

where a_1, a_2 are constants & Q is a function of x or simply constant.

We can write it as:

$$\left(\frac{d^2}{dx^2} + a_1 \frac{d}{dx} + a_2\right)y = Q$$

or $(D^2 + a_1 D + a_2)y = Q$ where $D = \frac{d}{dx}$

or $\{f(D)\}y = Q$ — (1)

where $f(D) = D^2 + a_1 D + a_2$.

Its solution is given by

$$y = C.F. + P.I.$$

where C.F. is complementary function and P.I. is particular integral.

Rules to find C.F. (Complementary function) -

- ① First, find the auxiliary equation A.E. by replacing D by m , y by 1 and Q by 0 in (1), i.e. $f(m) = 0$.
- ② Solve this to get the values of m (or the roots of m).
- ③ ~~At~~ Three cases arise:

(a) If the roots are distinct (i.e. $m_1 \neq m_2$), (42)

$$\text{then C.F.} = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

(b) If the roots are equal (i.e. $m_1 = m_2$), then

$$\text{C.F.} = (c_1 + c_2 x) e^{m_1 x}$$

(c) If the roots are imaginary, i.e.

$$m = \alpha \pm i\beta, \text{ then}$$

$$\text{C.F.} = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x).$$

e.g. If $m = 2, -5$, then

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{-5x}.$$

If $m = 2, 2$, then

$$\text{C.F.} = (c_1 + c_2 x) e^{2x}$$

If $m = 2 \pm 3i$, then

$$\text{C.F.} = e^{2x} (c_1 \cos 3x + c_2 \sin 3x).$$

Rules for Particular Integral:

$$\text{P.I.} = \frac{1}{f(D)} \cdot Q.$$

① If $Q = 0$, then P.I. = 0.

② If $Q = e^{ax}$ then P.I. = $\frac{1}{f(a)} e^{ax}$

③ If $Q = \sin ax$ or $\cos bx$, then

$$\text{P.I.} = \frac{1}{f(-a^2)} (\sin ax) \text{ or } (\cos bx)$$

Q. Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 0$.

Solution. The given differential equation is

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 0$$

$$\text{or } D^2y + 5Dy + 4y = 0$$

$$\text{or } (D^2 + 5D + 4)y = 0$$

\therefore Auxiliary equation is

$$m^2 + 5m + 4 = 0$$

$$\Rightarrow m^2 + 4m + m + 4 = 0$$

$$\Rightarrow m(m+4) + 1(m+4) = 0$$

$$\text{or } (m+4)(m+1) = 0$$

~~$$\Rightarrow m = -4$$~~

$$\text{or } (m+4) = 0 \text{ or } m+1 = 0$$

$$\Rightarrow m = -4 \text{ or } m = -1.$$

\therefore C.F. is

$$C.F. = c_1 e^{-4x} + c_2 e^{-x}$$

$$\text{Now, P.I.} = \frac{1}{f(D)} \cdot Q$$

$$= \frac{1}{D^2 + 5D + 4} \cdot (0)$$

$$= 0$$

Hence, the complete solution is

$$y = C.F. + P.I.$$

$$\text{or } y = c_1 e^{-4x} + c_2 e^{-x} \quad \leftarrow \text{Ans.}$$

Q. Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$.

(44)

Solution. The given differential equation is

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$$

$$\text{or } D^2y - 3Dy + 2y = e^{5x}$$

$$\text{or } (D^2 - 3D + 2)y = e^{5x}$$

\therefore A.E. is

$$m^2 - 3m + 2 = 0$$

$$\text{or } m^2 - 2m - m + 2 = 0$$

$$\text{or } m(m-2) - 1(m-2) = 0$$

$$\text{or } (m-2)(m-1) = 0$$

$$\text{or } m = 2, 1$$

$$\therefore \text{C.F.} = c_1 e^{2x} + c_2 e^x$$

$$\text{Now, P.I.} = \frac{1}{f(D)} \cdot Q$$

$$= \frac{1}{D^2 - 3D + 2} e^{5x}$$

$$= \frac{1}{5^2 - 3(5) + 2} e^{5x}$$

$$= \frac{1}{25 - 15 + 2} e^{5x} = \frac{1}{12} e^{5x}$$

Hence, the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{2x} + c_2 e^x + \frac{1}{12} e^{5x} \leftarrow \text{Ans.}$$

Q. Solve $\frac{d^2y}{dx^2} - y = \sin 2x$.

Solution. Given differential equation is

$$\frac{d^2y}{dx^2} - y = \sin 2x$$

$$\text{or } D^2y - y = \sin 2x$$

$$\text{or } (D^2 - 1)y = \sin 2x$$

∴ A. E. is

$$m^2 - 1 = 0$$

$$\text{or } m^2 = 1$$

$$\text{or } m = \sqrt{1} = \pm 1$$

$$\text{or } m = 1, -1.$$

$$\therefore \text{C.F.} = c_1 e^x + c_2 e^{-x}$$

$$\text{Now, P.I.} = \frac{1}{f(D)} \cdot Q$$

$$= \frac{1}{D^2 - 1} \cdot \sin 2x$$

$$= \frac{1}{(-2^2) - 1} \sin 2x$$

$$= \frac{1}{-4 - 1} \sin 2x$$

$$= -\frac{1}{5} \sin 2x.$$

Hence, the complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\text{or } y = c_1 e^x + c_2 e^{-x} - \frac{1}{5} \sin 2x. \leftarrow \text{Ans.}$$

Q. Solve $\frac{d^2y}{dx^2} - 4y = e^x + \sin 2x$

Solution. $(D^2 - 4)y = e^x + \sin 2x.$

∴ A.E. is

$m^2 - 4 = 0$

or $m^2 = 4$

or $m = \pm \sqrt{4}$

or $m = \pm 2$

or $m = 2, -2.$

∴ C.F. = $c_1 e^{2x} + c_2 e^{-2x}$

Now, P.I. = $\frac{1}{f(D)} \cdot Q$

= $\frac{1}{D^2 - 4} (e^x + \sin 2x)$

= $\frac{1}{D^2 - 4} (e^x) + \frac{1}{D^2 - 4} (\sin 2x)$

= $\frac{1}{1^2 - 4} e^x + \frac{1}{-2^2 - 4} \sin 2x$

= $-\frac{1}{3} e^x - \frac{1}{8} \sin 2x$

∴ The complete solution is

$y = C.F. + P.I.$

= $c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3} e^x - \frac{1}{8} \sin 2x$

Ans.