M.Sc. II Sem. (Mathematics)

Paper 1st - Advanced Abstract Algebra-II

Unit - V

Reference Book : • **I.N. Herstein, Topics in Algebra, Wiley Easter Ltd., New Delhi, 1975.**

Definition. If $T \in A(V)$, then $\lambda \in F$ is called a characteristic root (or eigen value) of T if $\lambda - T$ is regular.

Theorem. The element $\lambda \in F$ is a characteristic root of $T \in A(V)$ if and only if for some $v \neq 0$ in V, $vT = \lambda v$.

Proof. If λ is a characteristic root of T, then $\lambda - T$ is singular.

Hence, by the theorem, "If V is finite dimensional over F, then $T \in A(V)$ is singular if and only if there exists a $v \neq 0$ in V such that $vT = 0$", we get a vector $v \neq 0$ in V such that

$$v(\lambda - T) = 0$$

$$\Rightarrow \quad v\lambda - vT = 0$$

or $\quad v\lambda = vT$

or $\quad vT = v\lambda.$

On the other hand, if $vT = v\lambda$ for some $v \neq 0$ in V, then

$$-vT + v\lambda = 0$$

$$\Rightarrow \quad v(\lambda - T) = 0.$$

Hence, by the theorem, "If V is finite dimensional over F, then T $\in$ A(V) is singular if and only if there exists a $v \neq 0$ in V such that $vT = 0$",

$(\lambda - T)$ must be singular.

Hence, λ is a characteristic root of T.

Lemma. If $\lambda \in F$ is a characteristic root of $T \in A(V)$, then for any polynomial $q(x) \in F[x]$, $q(\lambda)$ is a characteristic root of $q(T)$.

Proof. Suppose that $\lambda \in F$ is a characteristic root of T. Then by above theorem, there is a non-zero vector v in V such that

$$vT = \lambda v.$$

Now, $vT^2 = (\lambda v)T$

$= \lambda(vT)$ [Since T is linear transformation]

$= \lambda(\lambda v)$ [Since vT = v]

$= \lambda^2 v$

i.e. $vT^2 = \lambda^2 v.$

Continuing in this way,

$vT^k = \lambda^k v$, for all positive integers k.

Suppose $q(x) = \alpha_0 x^m + \alpha_1 x^{m-1} + \ldots + \alpha_m, \quad \alpha_i \in F.$ (1)

Then, $q(T) = \alpha_0 T^m + \alpha_1 T^{m-1} + \ldots + \alpha_m.$

$\Rightarrow \quad v.q(T) = v(\alpha_0 T^m + \alpha_1 T^{m-1} + \ldots + \alpha_m)$

$= \alpha_0(vT^m) + \alpha_1(vT^{m-1}) + \ldots + \alpha_m v.$

$= \alpha_0 \lambda^m v + \alpha_1 \lambda^{m-1} v + \ldots + \alpha_m v$ [Since $vT^k = \lambda^k v$]

$= v(\alpha_0 \lambda^m + \alpha_1 \lambda^{m-1} + \ldots + \alpha_m)$

$= q(\lambda)v$ (by 1)

i.e. $vq(T) = q(\lambda)v$

$\Rightarrow$ $q(\lambda)$ is a characteristic root of $q(T)$. [By above theorem]

Hence proved.

Theorem. If λ is characteristic root of T, then λ is a root of minimal polynomial of T. In particular, T has a finite number of characteristic roots in F.

Proof. As we know that if λ is a characteristic root of T, then for any polynomial q(x) over F, there exist a non zero vector v such that $vq(T) = q(\lambda)v$.

If we take q(x) as minimal polynomial of T then $q(T)=0$.

But then $vq(T) = q(\lambda)v$

$\Rightarrow$ $q(\lambda)v=0$.

As v is non zero, therefore, $q(\lambda) = 0$ i.e. λ is root of q(x).

Or, λ is root of minimal polynomial of T.

Since q(x) has only a finite number of roots in F, there can only be a finite number of characteristic roots of T in F.

Definition. The element $0 \neq v \in V$ is called a characteristic vector of T belonging to the characteristic root $\lambda \in F$ if $vT = \lambda v$.

Definition. The linear transformations S, T A(V) are said to be similar if there exists an invertible element $C \in A(V)$ such that $T = CSC^{-1}$.

Theorem. If v_1, v_2, …, v_n are different characteristic vectors belonging to distinct characteristic roots λ_1, λ_2, …, λ_n respectively, then $v_1, v_2, \ldots, v_k$ are linearly independent over F.

Proof. Suppose if possible v_1, v_2, …, v_n are linearly dependent over F, then there exist a relation $\beta_1 v_1 + \ldots + \beta_n v_n = 0$, where $\beta_1 + \ldots + \beta_n$ are all in F and not all of them are zero. In all such relations, there is one relation having as few non-zero coefficients as possible. By suitably renumbering the vectors, let us assume that this shortest relation be

$$\beta_1 v_1 + \ldots + \beta_k v_k = 0, \text{ where } \beta_1 \neq 0, \ldots, \beta_k \neq 0. \qquad \text{(i)}$$

Applying T on both sides and using $v_i T = \lambda_i v_i$ in (i) we get

$$\lambda_1 \beta_1 v_1 + \ldots + \lambda_k \beta_k v_k = 0. \qquad \text{(ii)}$$

Multiplying (i) by λ_1 and subtracting from (ii), we obtain

$$(\lambda_2 - \lambda_1)\beta_2 v_2 + \ldots + (\lambda_k - \lambda_1)\beta_k v_k = 0.$$

Now $(\lambda_i - \lambda_1) \neq 0$ for $i > 1$ and $\beta_2 \neq 0$, therefore, $(\lambda_i - \lambda_1)\beta_i \neq 0$. But then we obtain a shorter relation than that in (i) between $v_1, v_2, \ldots, v_n$. This contradiction proves the theorem.

Corollary. If $\dim F_V = n$, then $T \in A(V)$ can have at most n distinct characteristic roots in F.

Proof. Suppose if possible T has more than n distinct characteristic roots in F, then there will be more than n distinct characteristic vectors belonging to these distinct characteristic roots. By above Theorem, these vectors will be linearly independent over F.

Since $\dim_F V = n$, these n+1 element will be linearly dependent, a contradiction. This contradiction proves T can have at most n distinct characteristic roots in F.