M.Sc. II Sem. (Mathematics)

Paper 1st - Advanced Abstract Algebra-II

Unit - V

Reference Book : • **I.N. Herstein, Topics in Algebra, Wiley Easter Ltd., New Delhi, 1975.**

Definition. The subspace W of V is invariant under $T \in A(V)$ if $WT \subset W$.

Remark. If p(x) is a minimal polynomial for T over F and if T satisfies a polynomial h(x), then $p(x) \mid h(x)$.

Theorem 1. If $W \subset V$ is invariant under T, then T induces a linear transformation $\bar{T}$ on V/W, defined by $(v + W)\bar{T} = vT + W$. If T satisfies the polynomial $q(x) \in F[x]$, then so does $\bar{T}$. If $p_1(x)$ is the minimal polynomial for $\bar{T}$ over F and if p(x) is that for T, then $p_1(x) \mid p(x)$.

Proof. Suppose $\bar{V} = V/W$ then the elements of $\bar{V}$ are the cosets of v + W of W in V.

Let $\bar{v} = v + W \in \bar{V}$ which is defined as

$$\bar{v}\,\bar{T} = vT + W.$$

To show that $\bar{T}$ is a linear transformation, firstly we show that $\bar{T}$ is well-defined on $\bar{V}$.

Suppose $\bar{v} = v_1 + W = v_2 + W$, where $v_1, v_2 \in V$.

Now, we show that $v_1T + W = v_2T + W$.

Since $v_1 + W = v_2 + W$, then

$$(v_1 - v_2) + W = W$$

$\Rightarrow \quad (v_1 - v_2) \in W.$

Since W is invariant under T and so

$$WT \subset W.$$

And so $(v_1 - v_2)T \in W$

$\Rightarrow \quad (v_1T - v_2T) \in W$ [Since T is linear transformation]

$\Rightarrow \quad (v_1T - v_2T) + W = W$

$\Rightarrow \quad v_1T + W = v_2T + W$

$\Rightarrow \quad \overline{T}$ is well-defined on $\overline{V}$.

Now, we prove that $\overline{T}$ is a linear transformation.

(i) We show that

$$\{(v_1 + W) + (v_2 + W)\}\overline{T} = (v_1 + W)\overline{T} + (v_2 + W)\overline{T},$$

where $v_1 + W, v_2 + W \in \overline{V}$

Consider

$\{(v_1 + W) + (v_2 + W)\}\overline{T} = \{(v_1 + v_2) + W\}\overline{T}$

[By definition of addition in quotient sets]

$= (v_1 + v_2)T + W$ [Be definition of $\overline{T}$]

$= (v_1T + v_2T) + W$ [Since T is linear transformation]

$= (v_1T + W) + (v_2T + W)$

[By definition of addition in quotient sets]

$= (v_1 + W)\overline{T} + (v_2 + W)\overline{T}$ [By definition of $\overline{T}$]

i.e. $\{(v_1 + W) + (v_2 + W)\}\overline{T} = (v_1 + W)\overline{T} + (v_2 + W)\overline{T}$.

(ii) Now, we prove that

$\{c(v + W)\}\overline{T} = c\{(v + W)\overline{T}\}$.

Consider

$\{c(v + W)\}\overline{T} = (cv + W)\overline{T}$ [By definition of scalar multiplication]

$= cvT + W$ [By definition of $\overline{T}$]

$= c(vT + W)$ [Since T is linear]

$= c((v + W)\overline{T})$ [By definition of $\overline{T}$]

i.e. $\{c(v + W)\}\overline{T} = c\{(v + W)\overline{T}\}$.

Hence, $\overline{T}$ is a linear transformation on $\overline{V}$.

Now, $\overline{v} = v + W \in \overline{V}$.

Then $\overline{v}(\overline{T}^2) = vT^2 + W$ [By definition]

$= (vT)T + W$

$= (vT + W)\overline{T}$

$= (v + W)\ \overline{T}\ \overline{T}$ [By definition of $\overline{T}$]

$= (v + W)(\overline{T})^2$

$= \overline{v}(\overline{T})^2$.

i.e. $\bar{v}(\bar{T}^2) = \bar{v}(\bar{T})^2$.

$\Rightarrow$ $(\bar{T}^2) = (\bar{T})^2$.

Similarly, $(\bar{T}^k) = (\bar{T})^k$ for any $k \geq 0$.

Consequently, for any polynomial $q(x) \in F[x]$,

$$\overline{q(T)} = q(\bar{T}).$$

For any $q(x) \in F[x]$ with $q(T) = 0$, since $\bar{0}$ is zero transformation on $\bar{V}$,

$$0 = \overline{q(T)} = q(\bar{T}).$$

Suppose $p_1(x)$ be the minimal polynomial over F satisfied by $\bar{T}$.

If $q(\bar{T}) = 0$, for $q(x) \in F[x]$, then $p_1(x) \mid q(x)$.

If $p(x)$ is the minimal polynomial for T over F, then $p(T) = 0$,

hence $p(\bar{T}) = 0$ and so $p_1(x) \mid p(x)$.

Hence proved.