M.Sc. II Sem. (Mathematics)

Paper 1st - Advanced Abstract Algebra-II

Unit - V

Reference Book :

- **P.B.Bhattacharya, S.K. Jain and S.R Nagpaul, *Basic Abstract Algebra* (2nd Edition), Cambridge University Press, Indian Edition, 1997.**
- **C. Musili, *Introduction to Rings and Modules,* Second Revised Edition, Narosa Publishing House, New Delhi.**
- **I.N. Herstein, Topics in Algebra, Wiley Easter Ltd., New Delhi, 1975.**

Definition. Given $T \in A(V)$. A non-trivial polynomial $p(x) \in F[x]$ of lowest degree with the property that $p(T) = 0$ is called a minimal polynomial for T over the field F.

Theorem 1. If V is finite dimensional over F, then $T \in A(V)$ is invertible if and only if the constant term of the minimal polynomial for T is not 0.

Proof. Suppose $p(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_k x^k$, $\alpha_k \neq 0$ be the minimal polynomial for T over F. Then $p(T) = 0$.

I Part : Suppose the constant term of p(x), i.e. $\alpha_0 \neq 0$. Then we have to show that $T \in A(V)$ is invertible..

Since $p(T) = 0$ and so

$$\alpha_k T^k + \alpha_{k-1} T^{k-1} + \ldots + \alpha_1 T + \alpha_0 = 0$$

or $$\alpha_0 = -\left(\alpha_k T^k + \alpha_{k-1} T^{k-1} + \ldots + \alpha_1 T\right)$$

or $$1 = -\frac{1}{\alpha_0}\left(\alpha_k T^k + \alpha_{k-1} T^{k-1} + \ldots + \alpha_1 T\right)$$

or $$1 = \left\{-\frac{1}{\alpha_0}\left(\alpha_k T^{k-1} + \alpha_{k-1} T^{k-2} + \ldots + \alpha_1\right)\right\} T$$

or $$\left\{-\frac{1}{\alpha_0}\left(\alpha_k T^{k-1} + \alpha_{k-1} T^{k-2} + \ldots + \alpha_1\right)\right\} T = 1$$

or $$ST = 1, \quad \text{where } S = -\frac{1}{\alpha_0}\left(\alpha_k T^{k-1} + \alpha_{k-1} T^{k-2} + \ldots + \alpha_1\right)$$

or $$S = T^{-1}$$

i.e., S acts as inverse of T.

Hence, T is invertible.

II Part : Suppose that T is invertible. Then we have to show that constant term of the minimal polynomial p(x), i.e., $\alpha_0 \neq 0$.

Suppose on the contrary that $\alpha_0 = 0$. Then we have

$$\alpha_1 T + \alpha_1 T^2 + \ldots + \alpha_k T^k = 0$$

or $$\left(\alpha_1 + \alpha_1 T + \ldots + \alpha_k T^{k-1}\right) T = 0.$$

Multiplying the above equation by T^{-1} from the right, we have

$$\alpha_1 + \alpha_1 T + \ldots + \alpha_k T^{k-1} = 0,$$

Hence, T satisfies the polynomial $q(x) = \alpha_1 + \alpha_1 x + \ldots + \alpha_k x^{k-1}$ in F[x].

Now, deg(q(x)) < deg (p(x)), which is contradiction.

Hence, $\alpha_0 \neq 0$.

Hence proved.

Corollary 1. If V is finite dimensional over F and if T $\in$ A(V) is invertible, then T^{-1} is a polynomial expression in T over F.

Proof. Since T is invertible, and so by the above theorem 1,

$$\alpha_0 + \alpha_1 T + \ldots + \alpha_k T^k = 0 \text{ with } \alpha_0 \neq 0.$$

Then $$\alpha_0 = -\left(\alpha_1 T + \alpha_2 T^2 + \ldots + \alpha_k T^k\right)$$

or $$1 = -\frac{1}{\alpha_0}\left(\alpha_1 T + \alpha_2 T^2 + \ldots + \alpha_k T^k\right)$$

or $$T^{-1} = -\frac{1}{\alpha_0}\left(\alpha_1 + \alpha_2 T + \ldots + \alpha_k T^{k-1}\right).$$

Hence, T^{-1} is a polynomial expression in T over F.

Corollary 2. If V is finite dimensional over F and if T $\in$ A(V) is singular, then there exists an $S \neq 0$ in A(V) such that ST = TS = 0.

Proof. Since T is singular, i.e., not regular and so by the above theorem 1, the constant term of its minimal polynomial must be 0.

That is, $p(x) = \alpha_1 x + \ldots + \alpha_k x^k$.

Also, $p(T) = \alpha_1 T + \alpha_2 T^2 + \ldots + \alpha_k T^k$.

If $S = \alpha_1 + \alpha_2 T + \ldots + \alpha_k T^{k-1}$, then

$S \neq 0$ (since $\alpha_1 + \alpha_2 x + \ldots + \alpha_k x^{k-1}$ is of lower degree than p(x)) and

ST = TS = 0.

Corollary 3. If V is finite dimensional over F and if T $\in$ A(V) is right invertible, then it is invertible.

Proof. Suppose T is right invertible and so TU = 1 for some U $\in$ A(V).

We have to show that T is invertible.

Suppose on the contrary that T is singular, then by corollary 2, there exists an S $\neq$ 0 such that ST = TS = 0.

However, 0 = (ST)U = S(TU) = S1 = S $\neq$ 0, which is a contradiction.

Hence, T is regular.

Theorem 2. If V is finite dimensional over F, then T $\in$ A(V) is singular if and only if there exists a v $\neq$ 0 in V such that vT = 0.

Proof. By corollary 2, T is singular if and only if there is an S $\neq$ 0 in A(V) such that ST = TS = 0.

Since S $\neq$ 0 there is an element w $\in$ V such that wS $\neq$ 0.

Let v = wS, then

$$vT = (ws)T = w(ST) = w0 = 0.$$

Hence proved.

Conversely, suppose that vT = 0 with v $\neq$ 0 in V.

Now, since S $\in$ A(V) and v $\neq$ 0 in V and so ST = 0.

Hence by corollary 2, T $\in$ A(V) is singular.