7.1.5 Scaling the Digital Transfer Function

After a digital filter has been designed following any one of the techniques outlined in this chapter, the corresponding transfer function $G(z)$ has to be scaled in magnitude before it can be implemented. In magnitude scaling, the transfer function is multiplied by a scaling constant K so that the maximum magnitude of the scaled transfer function $G_t(z) = K\,G(z)$ in the passband is unity, i.e., the scaled transfer function has a maximum gain of 0 dB. For a stable transfer function $G(z)$ with real coefficients, the scaled transfer function $K\,G(z)$ is then a bounded real (BR) function.[4]

For a frequency selective transfer function $G(z)$, if G_{max} is the maximum value of $|G(e^{j\omega})|$ in the frequency range $0 \le \omega \le \pi$, then $K = 1/G_{max}$, which results in a maximum gain of 0 dB in the passband of the scaled transfer function. For example, in the case of a lowpass transfer function with a maximum magnitude at dc, it is usual practice to use $K = 1/G(1)$, implying a dc gain of 0 dB for the scaled transfer function. Likewise, in the case of a highpass transfer function with a maximum magnitude at $\omega = \pi$, K is selected equal to $1/G(-1)$ yielding a gain of 0 dB at $\omega = \pi$ for the scaled transfer function. For a bandpass transfer function, it is common to use K equal to $1/|G(e^{j\omega_c})|$, where ω_c is the center frequency.

EXAMPLE 7.6 Consider the fourth-order highpass transfer function

$$G(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3} + p_4 z^{-4}}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3} + d_4 z^{-4}}.$$

Its magnitude scaled version is then given by

$$G_t(z) = K\,\frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3} + p_4 z^{-4}}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3} + d_4 z^{-4}},$$

where

$$K = \frac{1 - d_1 + d_2 - d_3 + d_4}{p_0 - p_1 + p_2 - p_3 + p_4}.$$

7.2 Bilinear Transformation Method of IIR Filter Design

A number of transformations has been proposed to convert an analog transfer function $H_a(s)$ into a digital transfer function $G(z)$ so that essential properties of the analog transfer function in the s-domain are preserved for the digital transfer function in the z-domain. Of these, the bilinear transformation is more commonly used to design IIR digital filters based on the conversion of analog prototype filters.

7.2.1 The Bilinear Transformation

The bilinear transformation from the s-plane to the z-plane is given by [Kai66]

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right). \tag{7.20}$$

The above transformation is a one-to-one mapping, i.e., it maps a single point in the s-plane to a unique point in the z-plane, and vice versa. The relation between the digital transfer function $G(z)$ and the parent analog transfer function $H_a(s)$ is then given by

$$G(z) = H_a(s)\big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}. \tag{7.21}$$

[4]See Section 4.4.5 for a definition of a bounded real (BR) function.

The bilinear transformation is derived by applying the trapezoidal numerical integration approach to the differential equation representation of $H_a(s)$ that leads to the difference equation representation of $G(z)$ (see Example 2.36). The parameter T represents the step size in the numerical integration. As we shall see later in this section, the digital filter design procedure consists of two steps: first, the inverse bilinear transformation is applied to the digital filter specifications to arrive at the specifications of the analog filter prototype; then the bilinear transformation of Eq. (7.20) is employed to obtain the desired digital transfer function $G(z)$ from the analog transfer function $H_a(s)$ designed to meet the analog filter specifications. As a result, the parameter T has no effect on the expression for $G(z)$, and we shall choose, for convenience, $T = 2$ to simplify the design procedure.

The corresponding inverse transformation for $T = 2$ is given by

$$z = \frac{1+s}{1-s}. \tag{7.22}$$

Let us now examine the above transformation. Note that for $s = j\Omega_o$,

$$z = \frac{1+j\Omega_o}{1-j\Omega_o}, \tag{7.23}$$

which has a unity magnitude. This implies that a point on the imaginary axis in the s-plane is mapped onto a point on the unit circle in the z-plane. In the general case, for $s = \sigma_o + j\Omega_o$,

$$z = \frac{1+(\sigma_o+j\Omega_o)}{1-(\sigma_o+j\Omega_o)} = \frac{(1+\sigma_o)+j\Omega_o}{(1-\sigma_o)-j\Omega_o}. \tag{7.24}$$

Therefore,

$$|z|^2 = \frac{(1+\sigma_o)^2+(\Omega_o)^2}{(1-\sigma_o)^2+(\Omega_o)^2}. \tag{7.25}$$

Thus, a point on the $j\Omega$-axis in the s-plane ($\sigma_o = 0$) is mapped onto a point on the unit circle in the z-plane as $|z| = 1$. A point in the left-half s-plane with $\sigma_o < 0$ is mapped onto a point inside the unit circle in the z-plane as $|z| < 1$. Likewise, a point in the right-half s-plane with $\sigma_o > 0$ is mapped onto a point outside the unit circle in the z-plane, as $|z| > 1$. Any point in the s-plane is mapped onto a unique point in the z-plane and vice versa. The mapping of the s-plane into the z-plane via the bilinear transformation is illustrated in Figure 7.3 and is seen to have all the desired properties. Also, there is no aliasing due to the one-to-one mapping.

The exact relation between the imaginary axis in the s-plane ($s = j\Omega$) and the unit circle in the z-plane ($z = e^{j\omega}$) is of interest. From Eq. (7.20) with $T = 2$ it follows that

$$j\Omega = \frac{1-e^{-j\omega}}{1+e^{-j\omega}} = j\tan\left(\frac{\omega}{2}\right),$$

or

$$\Omega = \tan\left(\frac{\omega}{2}\right), \tag{7.26}$$

which has been plotted in Figure 7.4. Note from this plot that the positive (negative) imaginary axis in the s-plane is mapped into the upper (lower) half of the unit circle in the z-plane. However, it is clear that the mapping is highly nonlinear since the complete negative imaginary axis in the s-plane from $\Omega = -\infty$ to $\Omega = 0$ is mapped into the lower half of the unit circle from $\omega = -\pi$ (i.e., $z = -1$) to $\omega = 0$ (i.e., $z = +1$), and the complete positive imaginary axis in the s-plane from $\Omega = 0$ to $\Omega = +\infty$ is mapped into the upper half of the unit circle from $\omega = 0$ (i.e., $z = +1$) to $\omega = +\pi$ (i.e., $z = -1$). This introduces a distortion in the frequency axis called *frequency warping*. The effect of warping is more evident in

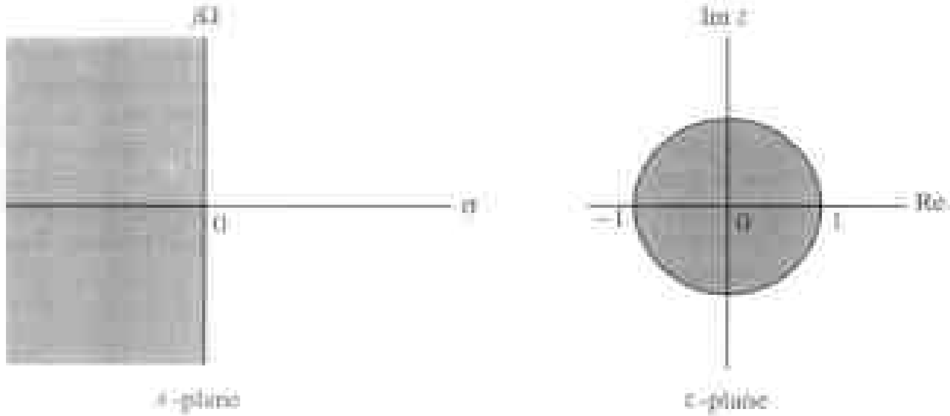

Figure 7.3: The bilinear transformation mapping.

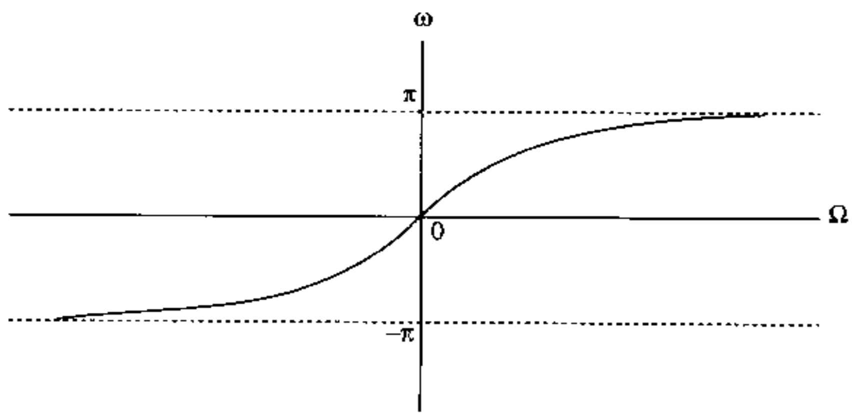

Figure 7.4: Mapping of the angular analog frequencies Ω to the angular digital frequencies ω via the bilinear transformation.

Figure 7.5, which shows the transformation of a typical analog filter magnitude response to a digital filter magnitude response derived via the bilinear transformation. Thus, to develop a digital filter meeting a specified magnitude response, we must first prewarp the critical bandedge frequencies (ω_p and ω_s) to find their analog equivalents (Ω_p and Ω_s) using the relation of Eq. (7.26), design the analog prototype $H_a(s)$ using the prewarped critical frequencies, and then transform $H_a(s)$ using the bilinear transformation to obtain the desired digital filter transfer function $G(z)$.

It should be noted that the bilinear transformation preserves the magnitude response of an analog filter only if the specification requires piecewise constant magnitude. However, the phase response of the analog filter is not preserved after transformation. Hence, the transformation can be used only to design digital filters with prescribed magnitude response with piecewise constant values.

EXAMPLE 7.7 It follows from Eqs. (5.34) and (5.35) that a first-order Butterworth lowpass transfer function with a 3-dB cutoff frequency at Ω_c is given by

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c}. \tag{7.27}$$