Properties of Z-Transform

Objective: To understand the properties of Z-Transform and associating the knowledge of properties of ROC in response to different operations on discrete signals.

Introduction :

We are aware that the z transform of a discrete signal *x(n)* is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

And inverse z transform is given by

$$x(n) = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

The Properties of z-transform simplifies the work of finding the z-domain equivalent of a time domain function when different operations are performed on discrete signal like time shifting, time scaling, time reversal etc. These properties also signify the change in ROC because of these operations.

These properties are also used in applying z- transform to the analysis and characterization of Discrete Time LTI systems.

Description :

1. **Linearity**

   Statement:

   If $x_1(n) \overset{Z}{\leftrightarrow} X_1(z)$ with ROC = R_1

   and $x_2(n) \overset{Z}{\leftrightarrow} X_2(z)$ with ROC = R_2

   then $ax_1(n) + bx_2(n) \overset{Z}{\leftrightarrow} aX_1(z) + bX_2(z)$, with ROC containing $R_1 \cap R_2$

   Proof:

   Taking the z-transform

$$Z\{ax_1(n) + bx_2(n)\} = \sum_{n=-\infty}^{\infty} \{ax_1(n) + bx_2(n)\}z^{-n}$$

$$= a\sum_{n=-\infty}^{\infty} x_1(n)z^{-n} + b\sum_{n=-\infty}^{\infty} x_2(n)z^{-n}$$

$$= aX_1(z) + bX_2(z)$$

The ROC of the Linear combination is at least the intersection of R_1 and R_2. For sequences with rational z-transforms, if the poles of $aX_1(z) + bX_2(z)$ consist of all the poles of $X_1(z)$ and $X_2(z)$, indicating no pole-zero cancellation, then the ROC will be exactly equal to the overlap of the individual regions of convergence.

If the Linear combination is such that some zeros are introduced that cancel poles, then the ROC may be larger.

Illustration:

A simple example of this occurs when $x_1(n)$ and $x_2(n)$ are both of infinite duration, but the linear combination is of finite duration. In this case the ROC of the linear combination is the entire z-plane, except for zero and / or infinity.

For example, the sequences $a^n u(n)$ and $a^n u(n-1)$ both have an ROC defined by $|z|>|a|$, but the sequence corresponding to the difference $\{a^n u(n) - a^n u(n-1)\}=\delta(n)$ has a region of convergence that is the entire z-plane.

2. Time Shifting

Statement:

If $x(n) \overset{Z}{\leftrightarrow} X(z)$ with ROC= R

then $x(n-m) \overset{Z}{\leftrightarrow} z^{-m}X(z)$ with ROC= R, except for the possible addition or deletion of the origin or infinity

Proof:

$$Z\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m)z^{-n}$$

Let n-m=p

$$= \sum_{p=-\infty}^{\infty} x(p)z^{-(p+m)}$$

$$= z^{-m} \sum_{p=-\infty}^{\infty} x(p)z^{-p}$$

$$= z^{-m}X(z)$$

Illustration:

Because of the multiplication by z^{-m}, for m>0 poles will be introduced at z=0, which may cancel corresponding zeros of X(z) at z=0. Consequently, z=0 may be a pole of $z^{-m}X(z)$ while it may not be a pole of X(z).In this case the ROC for $z^{-m}X(z)$ equals the ROC of X(z) but with the origin deleted.

Similarly, if m<0, zeros will be introduced at z=0, which may cancel corresponding poles of X(z) at z=0. Consequently, z=0 may be a zero of $z^{-m}X(z)$ while it may not be a pole of X(z). In this case z=∞ is a pole of $z^{-m}X(z)$, and thus the ROC for $z^{-m}X(z)$ equals the ROC of X(z) but with z=∞ deleted.

3. Scaling in the z-Domain

Statement:

If $x(n) \overset{Z}{\leftrightarrow} X(z)$ with ROC= R

then $z_o^{\,n}x(n) \overset{Z}{\leftrightarrow} X\left(\frac{z}{z_o}\right)$ with ROC= $|z_o|$R where, $|z_o|$R is the scaled version of R.

Proof:

$$Z\{z_o{}^n x(n)\} = \sum_{n=-\infty}^{\infty} z_o{}^n x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n)\left(\frac{z}{z_o}\right)^{-n} = X\left(\frac{z}{z_o}\right)$$

Illustration:

If z is a point in the ROC of X(z), then the point $|z_o|z$is in the ROC of $X\left(\frac{z}{z_o}\right)$.

Also, if X(z) has a pole (or zero) at $z=a$, then $X\left(\frac{z}{z_o}\right)$has a pole (or zero) at $z=z_oa$.

An important special case of the property is when $z_o=e^{j\omega o}$. In this case, $|z_o|R=R$ and

$$e^{j\omega_o n}x(n) \overset{Z}{\leftrightarrow} X(e^{-j\omega_o}z)$$

The left-hand side of the above equation corresponds to multiplication by a complex exponential sequence. The right-hand side can be interpreted as a rotation in the z-plane; i.e., all pole-zero locations rotate in the z-plane by an angle of ω_o, as illustrated in the figure below.

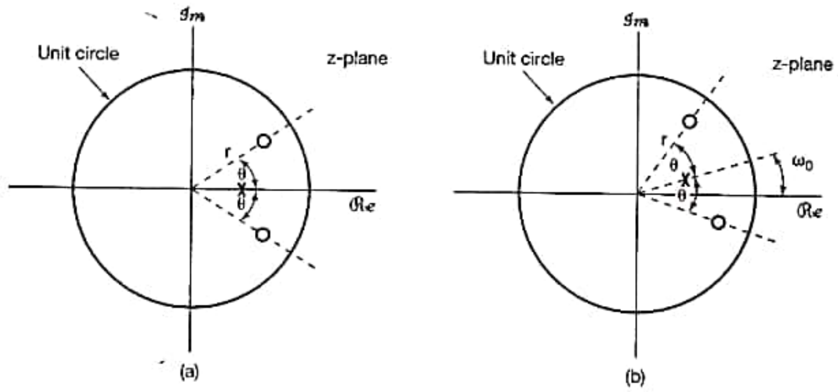

Fig (a) is the pole-zero pattern for the z-transform for a signal x(n)

Fig (b) is the pole-zero pattern for the z-transform of $e^{j\omega_o n}x(n)$

4. Time Reversal

Statement:

If $x(n) \overset{Z}{\leftrightarrow} X(z)$ with ROC= R

then $x(-n) \overset{Z}{\leftrightarrow} X\left(\frac{1}{z}\right)$ with ROC= $\frac{1}{R}$

Proof:

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)z^{-n}$$

Let -n=p

$$= \sum_{p=-\infty}^{\infty} x(p)(z)^{p} = \sum_{p=-\infty}^{\infty} x(p)(z^{-1})^{-p} = X\left(\frac{1}{z}\right)$$

Illustration

If z_o is in the ROC for x(n), then $1/z_o$ is in the ROC for x(-n)

5. Conjugation

Statement:

If $x(n) \overset{Z}{\leftrightarrow} X(z)$ with ROC= R

then $x^*(n) \overset{Z}{\leftrightarrow} X^*(z^*)$ with ROC= R

Proof:

$$Z\{x^*(n)\} = \sum_{n=-\infty}^{\infty} x^*(n)z^{-n}$$

as we know that $z=re^{j\omega}$

$$= \sum_{n=-\infty}^{\infty} x^*(n)r^{-n}\, e^{-j\omega n}$$

$$= \left(\sum_{n=-\infty}^{\infty} x(n)r^{-n}\, e^{+j\omega n}\right)^*$$

$$= \left(\sum_{n=-\infty}^{\infty} x(n)\,(z^*)^{-n}\right)^*$$

$$= \left(X(z^*)\right)^* = X^*(z^*)$$

Also $X(z) = X^*(z^*)$ when x(n) is real.

Illustration:

If X(z) has a pole (or zero) at $z=z_o$, it must also have a pole (or zero) at the complex conjugate point $z=z_o^*$.

6. The Convolution Property

Statement:

If $x_1(n) \overset{Z}{\leftrightarrow} X_1(z)$ with ROC = R_1

and $x_2(n) \overset{Z}{\leftrightarrow} X_2(z)$ with ROC = R_2

then $x_1(n) * x_2(n) \overset{Z}{\leftrightarrow} X_1(z).X_2(z)$, with ROC containing $R_1 \cap R_2$

Proof:

$$Z\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{\infty} \{x_1(n) * x_2(n)\}z^{-n}$$

$$= \sum_{n=-\infty}^{\infty}\left\{\sum_{m=-\infty}^{\infty} x_1(m)\, x_2(n-m)\right\}z^{-n}$$

Interchanging the order of summations

$$Z\{x_1(n) * x_2(n)\} = \sum_{m=-\infty}^{\infty} x_1(m)\left\{\sum_{n=-\infty}^{\infty} x(n-m)\, z^{-n}\right\}$$

$$= \sum_{m=-\infty}^{\infty} x_1(m)\{z^{-m} X_2(z)\}$$

(Since from Time shifting property)

$$= X_2(z)\left\{\sum_{m=-\infty}^{\infty} x_1(m) z^{-m}\right\}$$

$$= X_1(z).X_2(z)$$

Illustration:

Just as with the convolution property for the Laplace transform, the ROC of $X_1(z).X_2(z)$ includes the intersection of R_1 and R_2 and may be larger if pole-zero cancellation occurs in the product.

Note: This property plays an important role in the analysis of Discrete Time LTI systems.

For example consider an LTI system for which y(n)=h(n)*x(n), where $h(n) = \delta(n) - \delta(n-1)$.

Note that $\delta(n) - \delta(n-1) \overset{Z}{\leftrightarrow} 1 - z^{-1}$, with ROC equal to the entire z-plane except the origin. Also, the z-transform has a zero at z=1.

Applying the property

If $x(n) \overset{Z}{\leftrightarrow} X(z)$ with ROC = R, then
$y(n) \overset{Z}{\leftrightarrow} (1 - z^{-1})\, X(z)$ with ROC = R, with the possible deletion of z=0 and/or addition of z=1.