The z-transform

See Oppenheim and Schafer, Second Edition pages 94–139, or First Edition pages 149–201.

1 Introduction

The z-transform of a sequence $x[n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.$$

The z-transform can also be thought of as an operator $\mathcal{Z}\{\cdot\}$ that transforms a sequence to a function:

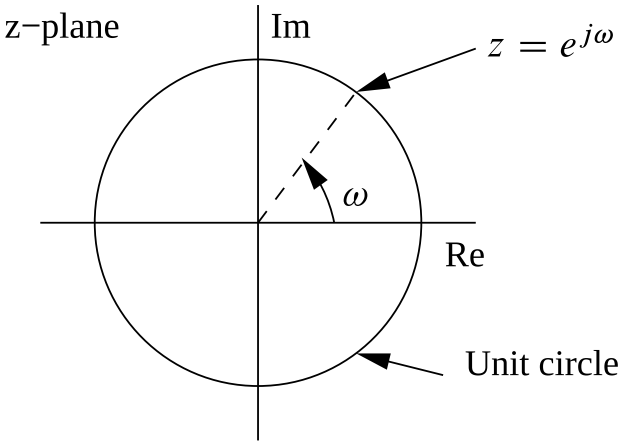
$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z).$$

In both cases z is a continuous complex variable.

We may obtain the Fourier transform from the z-transform by making the substitution $z = e^{j\omega}$. This corresponds to restricting $|z| = 1$. Also, with $z = re^{j\omega}$,

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})\, e^{-j\omega n}.$$

That is, the z-transform is the Fourier transform of the sequence $x[n]r^{-n}$. For $r = 1$ this becomes the Fourier transform of $x[n]$. The Fourier transform therefore corresponds to the z-transform evaluated on the unit circle:



The inherent periodicity in frequency of the Fourier transform is captured naturally under this interpretation.

The Fourier transform does not converge for all sequences — the infinite sum may not always be finite. Similarly, the z-transform does not converge for all sequences or for all values of z. The set of values of z for which the z-transform converges is called the **region of convergence (ROC)**.

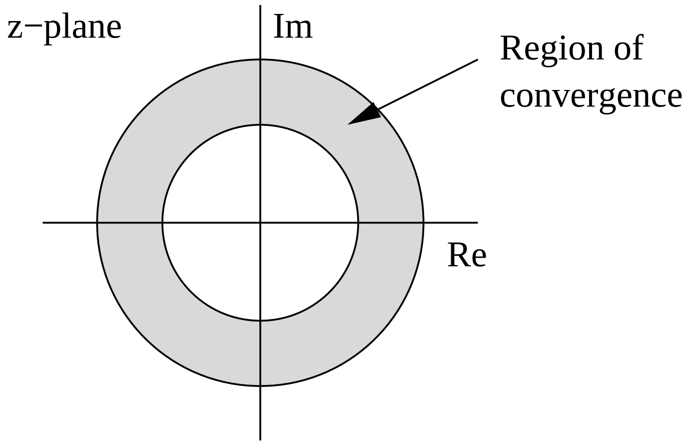
The Fourier transform of $x[n]$ exists if the sum $\sum_{n=-\infty}^{\infty} |x[n]|$ converges. However, the z-transform of $x[n]$ is just the Fourier transform of the sequence $x[n]r^{-n}$. The z-transform therefore exists (or converges) if

$$X(z) = \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty.$$

This leads to the condition

$$\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty$$

for the existence of the z-transform. The ROC therefore consists of a ring in the z-plane:



In specific cases the inner radius of this ring may include the origin, and the outer radius may extend to infinity. If the ROC includes the unit circle $|z| = 1$, then the Fourier transform will converge.

Most useful z-transforms can be expressed in the form

$$X(z) = \frac{P(z)}{Q(z)},$$

where $P(z)$ and $Q(z)$ are polynomials in z. The values of z for which $P(z) = 0$ are called the **zeros** of $X(z)$, and the values with $Q(z) = 0$ are called the **poles**. The zeros and poles completely specify $X(z)$ to within a multiplicative constant.

Example: right-sided exponential sequence
Consider the signal $x[n] = a^n u[n]$. This has the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$

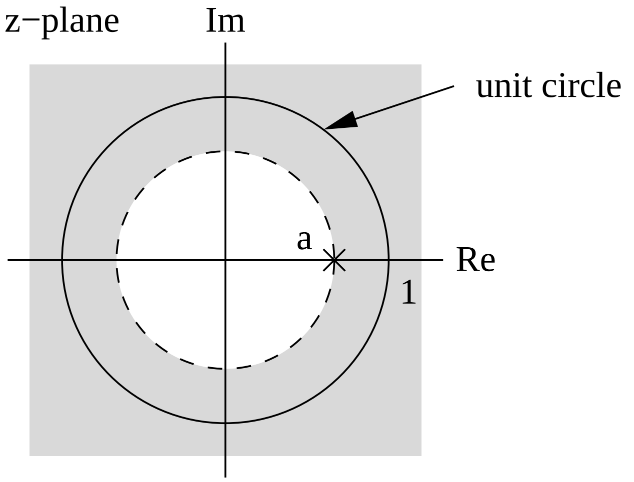
Convergence requires that

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty,$$

which is only the case if $|az^{-1}| < 1$, or equivalently $|z| > |a|$. In the ROC, the series converges to

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| > |a|,$$

since it is just a geometric series. The z-transform has a region of convergence for any finite value of a.



The Fourier transform of $x[n]$ only exists if the ROC includes the unit circle, which requires that $|a| < 1$. On the other hand, if $|a| > 1$ then the ROC does not include the unit circle, and the Fourier transform does not exist. This is consistent with the fact that for these values of a the sequence $a^n u[n]$ is exponentially growing, and the sum therefore does not converge.

Example: left-sided exponential sequence

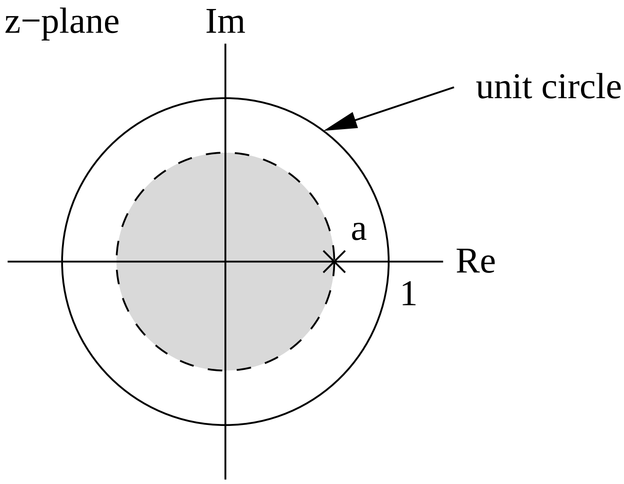
Now consider the sequence $x[n] = -a^n u[-n-1]$. This sequence is left-sided because it is nonzero only for $n \leq -1$. The z-transform is

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n.$$

For $|a^{-1}z| < 1$, or $|z| < |a|$, the series converges to

$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| < |a|.$$



Note that the expression for the z-transform (and the pole zero plot) is exactly the same as for the right-handed exponential sequence — *only the region of convergence is different.* Specifying the ROC is therefore critical when dealing with the z-transform.

Example: sum of two exponentials

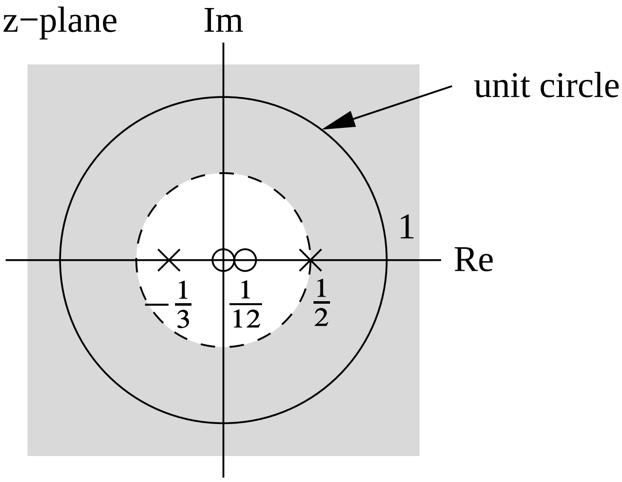
The signal $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$ is the sum of two real exponentials. The z-transform is

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \right\} z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n u[n] z^{-n} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n . \end{aligned}$$

From the example for the right-handed exponential sequence, the first term in this sum converges for $|z| > 1/2$, and the second for $|z| > 1/3$. The combined transform $X(z)$ therefore converges in the intersection of these regions, namely when $|z| > 1/2$. In this case

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}} = \frac{2z(z - \frac{1}{12})}{(z - \frac{1}{2})(z + \frac{1}{3})}.$$

The pole-zero plot and region of convergence of the signal is



Example: finite length sequence

The signal

$$x[n] = \begin{cases} a^n & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

has z-transform

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$
$$= \frac{1 - (az^{-1})^N}{1 - az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}.$$

Since there are only a finite number of nonzero terms the sum always converges when az^{-1} is finite. There are no restrictions on a ($|a| < \infty$), and the ROC is the entire z-plane with the exception of the origin $z = 0$ (where the terms in the sum are infinite). The N roots of the numerator polynomial are at

$$z_k = ae^{j(2\pi k/N)}, \qquad k = 0, 1, \ldots, N-1,$$